ON THE FOURIER EXPANSIONS OF JACOBI FORMS OF HALF-INTEGRAL WEIGHT

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Abstract. Using the relationship between Jacobi forms of half-integral weight and vector valued modular forms, we obtain the number of components which determine the given Jacobi form of index p, p^2 or pq, where p and q are odd primes.

1. Introduction

In his recent papers H. Skogman [4, 5] showed that when certain components of a vector-valued modular form associated to a Jacobi form of weight k (k is a positive integer) and index m are zero then some of the other components must be zero, where m is a square-free positive integer. More precisely, he showed that a Jacobi form of weight k and square-free index on the full Jacobi group is uniquely determined by any of the associated vector components.

In this article we generalize the work of Skogman to Jacobi forms of half-integral weight.

2. Preliminaries and Statement of Results

Let k, N be a positive integers and χ be a Dirichlet character modulo $4N$. Let $\mathcal{H}$ denote the complex upper half-plane. For a complex number z, let

$$\sqrt{z} = |z|^{1/2} e^{\frac{1}{2}\arg\, z}, \text{ with } -\pi < \arg\, z \leq \pi.$$
$$z^{k/2} = (\sqrt{z})^k \quad \text{for any } k \in \mathbb{Z}.$$

For $z \in \mathbb{C}$ and $a, b \in \mathbb{Z}$, we put $e_b^a(z) = e^{2\pi i a z/b}$. When a or b equals 1, we write $e_1^a(z) = e^a(z)$, $e_b^1(z) = e_b(z)$, and when $a = b = 1$, we write $e_1^1(z) = e(z)$. For $a, b \in \mathbb{Z}$, the symbol $a \pmod b$ means a set of congruent classes modulo b and $\gcd(a, b)$ means the greatest common divisor of a and b.

A *Jacobi form* $\phi(\tau, z)$ of weight $k + 1/2$ and index m for the group $\Gamma_0(4N)$, with character χ, is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following conditions.

(i)

$$\left(\frac{c}{d}\right)^{-1} \left(\frac{-4}{d}\right)^{k+1/2} (c\tau + d)^{-k-1/2} e^m \left(\frac{-c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)$$
$$\times\, \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right) \quad = \quad \chi(d)\phi(\tau, z),$$

2000 *Mathematics Subject Classification.* 11F50, 11F37, 11F30.

where $\tau \in \mathcal{H}$, $z \in \mathbb{C}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ and $\lambda, \mu \in \mathbb{Z}$ and $\left(\frac{c}{d}\right)$ denotes the Jacobi symbol.

(ii) For every $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$(c\tau+d)^{-k-1/2} e^m \left(\frac{-cz^2}{c\tau+d}\right) \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$$

has a Fourier development of the form

$$\sum_{\substack{n,r\in\mathbb{Q}\\ r^2\leq 4nm}} c_{\phi,\alpha}(n,r)\ e\left(n\tau+rz\right),$$

where the sum varies over rational numbers n, r with bounded denominators subject to the condition $r^2 \leq 4nm$.

Further, if $c_{\phi,\alpha}(n,r)$ satisfies the condition $c_{\phi,\alpha}(n,r) \neq 0$ implies $r^2 < 4nm$, then ϕ is called a *Jacobi cusp form*. We denote by $J_{k+1/2,m}(4N,\chi)$, the space of Jacobi forms of weight $k+1/2$, index m for $\Gamma_0(4N)$ with character χ. For other details, we refer to [3, 6].

Let $\phi(\tau,z)$ be a Jacobi form in $J_{k+1/2,m}(4N,\chi)$. Then its Fourier expansion at the infinite cusp is of the form

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z}\\ r^2\leq 4nm}} c_\phi(n,r) e(n\tau+rz). \tag{1}$$

The following property of the Fourier coefficients of a Jacobi form of half-integral weight follows easily as in the case of integral weight case (see [2]).

Lemma 2.1. *Let $\phi \in J_{k+1/2,m}(4N,\chi)$ having a Fourier expansion as in* (1)*. Then $c_\phi(n,r)$ depends only on $4mn - r^2$ and on $r \pmod{2m}$.*

For $D \leq 0$, $\mu \pmod{2m}$, define $c_\mu(|D|)$ as follows:

$$c_\mu(|D|) := \begin{cases} c_\phi(\frac{r^2-D}{4m}, r) & \text{if } D \equiv \mu^2 \pmod{4m},\ r \equiv \mu \pmod{2m}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

For $\mu \pmod{2m}$, let

$$h_\mu(\tau) := \sum_{|D|=0}^{\infty} c_\mu(|D|) e_{4m}\left(|D|\tau\right). \tag{3}$$

For $\alpha \pmod{2m}$, the Jacobi theta function is given by

$$\vartheta_\alpha(\tau,z) = \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\alpha \pmod{2m}}} e\left(\frac{r^2}{4m}\tau + rz\right). \tag{4}$$

By standard arguments (see [2, 4]), the Jacobi form $\phi \in J_{k+1/2,m}(4N,\chi)$ can be expressed as follows.

$$\phi(\tau,z)=\sum_{\mu \pmod{2m}} h_\mu(\tau)\vartheta_\mu(\tau,z). \tag{5}$$

We now state our results.

Theorem 2.2. *Let $\phi(\tau,z)\in J_{k+1/2,p}(4N,\chi)$, where p is an odd prime such that* $\gcd(N,2p)=1$. *For some $\alpha,\beta \pmod{2p}$ with $2\not|\alpha$, $2|\beta$ and* $\gcd(\alpha\beta,p)=1$, *if we have $h_\alpha(\tau)=0$ and $h_\beta(\tau)=0$, then $\phi(\tau,z)=0$.*

Theorem 2.3. *Let $\phi(\tau,z)\in J_{k+1/2,p}(4N,\chi)$, where p is an odd prime such that $p|N$. Then among the $2p$ components $h_\mu(\tau)$, λ_χ of them determine the Jacobi form $\phi(\tau,z)$, where*

$$\lambda_\chi=\begin{cases} p-1 & \text{if } \chi \text{ is odd,}\\ p+1 & \text{if } \chi \text{ is even.}\end{cases}$$

Theorem 2.4. *Let $\phi(\tau,z)\in J_{k+1/2,pq}(4N,\chi)$, where p,q are distinct odd primes. Then the components h_α and h_β with $2|\alpha$, $2\not|\beta$ and* $\gcd(\alpha\beta,pq)=1$ *determine the associated Jacobi form $\phi(\tau,z)$.*

Theorem 2.5. *Let $\phi(\tau,z)\in J_{k+1/2,p^2}(4N,\chi)$, where p is an odd prime such that* $\gcd(p,N)=1$. *Then among the $2p^2$ components $h_\mu(\tau)$, the following components*

$$\{h_a,h_b\}, 2|a, 2\not|b,\ \gcd(ab,p)=1 \text{ and } \left\{h_{2ip},h_{(2i+1)p}\right\}_{i=0}^{(p-1)/2},$$

when χ is even and

$$\{h_a,h_b\}, 2|a, 2\not|b,\ \gcd(ab,p)=1 \text{ and } \left\{h_{2(i+1)p}\right\}_{i=0}^{(p-3)/2}, \left\{h_{(2i-1)p}\right\}_{i=1}^{(p-1)/2},$$

when χ is odd determine the Jacobi form $\phi(\tau,z)$. In other words, at most $p+1$ or $p+3$ (according as χ is odd or even) of the components h_μ will determine the Jacobi form $\phi(\tau,z)$.

3. Proofs

We need the following lemma, which was proved by Y. Tanigawa [6, Lemma 3].

Lemma 3.1. *Let $\phi(\tau,z)\in J_{k+1/2,m}(4N,\chi)$ and let h_α, $\alpha \pmod{2m}$ be a component of ϕ as described in* (5). *Then the following formulas hold for $h_\alpha(\tau)$.*

(i)
$$h_{-\alpha}(\tau)=\chi(-1)h_\alpha(\tau).$$

(ii)
$$h_\alpha(\tau+b)e_{4m}(\alpha^2 b)=h_\alpha(\tau), \qquad \textit{for any } b\in\mathbb{Z}.$$

(iii)
$$(4N\tau+1)^k\ h_\alpha(\tau)=\sum_{\beta \pmod{2m}}\xi_{\alpha,\beta}\ h_\beta\left(\frac{\tau}{4N\tau+1}\right)$$

or equivalently

$$(-4N\tau+1)^{-k}\ h_\alpha\left(\frac{\tau}{-4N\tau+1}\right)=\sum_{\beta \pmod{2m}}\xi_{\alpha,\beta}\ h_\beta(\tau),$$

where

$$\xi_{\alpha,\beta} = \frac{1}{2m} \sum_{\gamma \pmod{2m}} e_{4m}(-4N\gamma^2 + 2\gamma(\beta - \alpha)). \tag{6}$$

Remark 3.1. The above lemma gives the relationship among the $h_\alpha(\tau)$. If one of the $h_\alpha(\tau)$ is zero, then there is a linear dependence equation among the $h_\alpha(\tau)$ which is given by the transformation rule (iii) of the Lemma. It is natural to ask about the maximum number of components $h_\alpha(\tau)$ that can be zero in order that the given form ϕ is non-zero. Our theorems are motivated by this question considered by Skogman for the Jacobi forms of integral weight. Note that if χ is an odd character, i.e., $\chi(-1) = -1$ then $h_0(\tau) = 0 = h_m(\tau)$, which follows from (i) of the above lemma.

Remark 3.2. Let $\phi(\tau, z) \in J_{k+1/2,1}(4N, \chi)$ be a Jacobi form of index 1. Using (i) of Lemma 3.1, we see that when χ is an odd character, then $\phi(\tau, z) = 0$. Therefore, there is no non-zero Jacobi form of weight $k + 1/2$ and index 1 when the character χ is odd.

Remark 3.3. Let $m = 2$. In this case the Jacobi form $\phi(\tau, z) \in J_{k+1/2,2}(4N, \chi)$ will have four components $h_\mu(\tau)$, $\mu = 0, 1, 2, 3$. When χ is an odd character, then using (i) of Lemma 3.1 we get $h_0 = 0 = h_2$ and $h_1 = -h_3$. Therefore, $h_1(\tau)$ or $h_3(\tau)$ determines $\phi(\tau, z) \in J_{k+1/2,2}(4N, \chi)$ when χ is odd. Now, let $\phi(\tau, z) \in J_{k+1/2,2}(4N, \chi)$, N odd and χ be an even character. In this case, $h_1 = h_3$. Also, it can be seen that $\xi_{0,1} = \xi_{0,3} = 0$ and $\xi_{0,2} = 1$. Using this in the transformation (iii) of Lemma 3.1 and substituting $\alpha = 0$ and assuming that $h_0 = 0$, we get

$$\begin{aligned} 0 &= \xi_{0,1} h_1(\tau) + \xi_{0,2} h_2(\tau) + \xi_{0,3} h_3(\tau) \\ &= 2\xi_{0,1} h_1(\tau) + \xi_{0,2} h_2(\tau) \\ &= h_2(\tau). \end{aligned}$$

Therefore, $h_0 = 0$ implies that $h_2 = 0$. Already we have seen that h_1 determines h_3. This shows that $h_0(\tau)$ and $h_1(\tau)$ determine $\phi(\tau, z)$ when $m = 2$, χ is even and N is odd.

Before we proceed to prove the theorems, we shall evaluate the Gauss sum in (6) when m is odd and $\gcd(m, N) = 1$.

Lemma 3.2. *Let m be an odd natural number such that* $\gcd(m, N) = 1$ *and let* $\xi_{\alpha,\beta}$ *be the Gauss sum as defined in* (6)*. Then*

$$\xi_{\alpha,\beta} = \begin{cases} \frac{1}{\sqrt{m}} \left(\frac{-N}{m}\right) \epsilon_m \; e_{4m}\left((4N)^{-1}(\beta - \alpha)^2\right) & \text{if } 2|(\beta - \alpha), \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

where $(4N)^{-1}$ *is an integer which is the inverse of* $4N$ *modulo* m *and* $\epsilon_m = 1$ *or* i *according as* $m \equiv 1$ *or* $3 \pmod 4$*.*

Proof. Since $\gcd(2, m) = 1$, we write the representatives $r(\bmod 2m)$ as $r = mr_1 + 2r_2$, where r_1 varies modulo 2 and r_2 varies modulo m. Then the Gauss sum simplifies to

$$\frac{1}{2m}\sum_{\substack{r_1 \pmod 2\\ r_2 \pmod m}} e_m\left(-4Nr_2^2 + r_2(\beta-\alpha)\right) e_2\left(r_1(\beta-\alpha)\right).$$

The sum modulo 2 is zero unless $2|(\beta-\alpha)$ and the sum over m is the standard quadratic Gauss sum if $\gcd(m,2N)=1$ (see for instance [1, p.195]). Therefore, if $2|(\beta-\alpha)$, we have

$$\begin{aligned}\xi_{\alpha,\beta} &= \frac{1}{m} e_{4m}\left((4N)^{-1}(\beta-\alpha)^2\right)\sum_{r \pmod m} e_m\left(-(4N)^{-1}r^2\right)\\ &= \frac{1}{\sqrt{m}}\left(\frac{-4N}{m}\right)\epsilon_m\, e_{4m}\left((4N)^{-1}(\beta-\alpha)^2\right)\\ &= \frac{1}{\sqrt{m}}\left(\frac{-N}{m}\right)\epsilon_m e_{4m}\left((4N)^{-1}(\beta-\alpha)^2\right),\end{aligned}$$

where $(4N)^{-1}$ denotes an integer such that $4N(4N)^{-1} \equiv 1 \pmod m$.

□

3.1. **Proof of Theorem 2.2.** We begin this section by observing some properties of $\xi_{\alpha,\beta}$. Using the definition of $\xi_{\alpha,\beta}$ it can be easily verified that

$$\begin{aligned}\xi_{\alpha,-\beta} &= \xi_{-\alpha,\beta},\\ \xi_{\alpha,\beta} &= \xi_{-\alpha,-\beta}.\end{aligned} \tag{8}$$

Assume that for an odd α, $0 \le \alpha \le 2p$ with $\gcd(\alpha,p)=1$ we have $h_\alpha(\tau)=0$. We will show that $h_\mu(\tau)=0$ for all odd $\mu \pmod{2p}$. By Lemma 3.1 (i) we have $h_{-\alpha}(\tau)=0$. Substituting $h_\alpha = 0 = h_{-\alpha}$ in the equivalent form of Lemma 3.1 (iii), we get

$$\begin{aligned}\sum_\beta \xi_{\alpha,\beta} h_\beta(\tau) &= 0,\\ \sum_\beta \xi_{-\alpha,\beta} h_\beta(\tau) &= 0.\end{aligned} \tag{9}$$

Observe that in the expansion of each of the $h_\beta(\tau)$, only $h_\beta(\tau)$ and $h_{-\beta}(\tau)$ have terms of the form $e\left(\frac{-\alpha^2}{4p}\tau\right) e(n\tau)$ (since $x \equiv \mu^2 \pmod{4p}$ has only two solutions $\pm\mu$ modulo $2p$, if $\gcd(\mu,p)=1$). Using this in the above inversion formulas, we get

$$\begin{aligned}\xi_{\alpha,\beta}h_\beta(\tau) + \xi_{\alpha,-\beta}h_{-\beta}(\tau) &= 0,\\ \xi_{-\alpha,\beta}h_\beta(\tau) + \xi_{-\alpha,-\beta}h_{-\beta}(\tau) &= 0.\end{aligned} \tag{10}$$

Hence,

$$h_\beta(\tau) = -\frac{\xi_{\alpha,-\beta}}{\xi_{\alpha,\beta}} h_{-\beta}(\tau) = -\frac{\xi_{-\alpha,-\beta}}{\xi_{-\alpha,\beta}} h_{-\beta}(\tau). \tag{11}$$

Claim 1. For $\beta \pmod{2p}$ with $\gcd(\beta,2p)=1$, we have

$$\xi_{-\alpha,\beta}\xi_{\alpha,-\beta} \neq \xi_{\alpha,\beta}\xi_{-\alpha,-\beta} \quad \text{for } \alpha \equiv \beta \pmod 2.$$

From this it follows that $h_{-\beta}(\tau) = 0 = h_\beta(\tau)$ when $\gcd(\beta, 2p) = 1$.

Proof of Claim 1. Assume the contrary, i.e., assume that

$$\xi_{-\alpha,\beta}\xi_{\alpha,-\beta} = \xi_{\alpha,\beta}\xi_{-\alpha,-\beta}.$$

Then it follows that

$$\xi^2_{\alpha,\beta} = \xi^2_{\alpha,-\beta},$$

from which we should have

$$\xi_{\alpha,\beta} = \pm\xi_{\alpha,-\beta}. \tag{12}$$

We shall now show that the last identity is not true when $\gcd(\beta, 2p) = 1$.

By Lemma 3.2, we get

$$\xi_{\alpha,-\beta} = \frac{1}{\sqrt{p}}\left(\frac{-N}{p}\right)\epsilon_p\ e_{4p}\left(N'(\alpha+\beta)^2\right),$$

if $2|(\alpha+\beta)$, where N' is an integer such that $(4N)N' \equiv 1 \pmod p$. Therefore, for (12) to be true we must have

$$e_p(-N'\alpha\beta) = \pm 1,$$

which implies that $p|N'\alpha\beta$, a contradiction. Similarly, if one starts with $h_\alpha = 0$ for even α, then $h_\beta = 0$ for all β even with $\gcd(\beta, p) = 1$. It remains to show that $h_0(\tau) = 0 = h_p(\tau)$. If χ is an odd character then as remarked in Remark 3.1, we have $h_0(\tau) = 0 = h_p(\tau)$. So, let χ be an even character. If $h_\alpha(\tau) = 0$ for an even α, then applying the inversion formula as before and observing that $x^2 \equiv 0 \pmod{4p}$ has only one solution $x = 0$ modulo $2p$, we have $\xi_{\alpha,0}h_0(\tau) = 0$. Since $\xi_{\alpha,0} \neq 0$, it follows that $h_0(\tau) = 0$. Similarly, one can prove that $h_p(\tau) = 0$ by considering the square class p^2 modulo $4p$ which has only one solution $(= p)$ modulo $2p$. This completes the proof of Theorem 2.2

3.2. **Proof of Theorem 2.3.** Here the index is p and $p|N$. In this case we have

$$\begin{aligned}\xi_{\alpha,\beta} &= \frac{1}{2p}\sum_{\gamma \pmod{2p}} e_{2p}\big(\gamma(\beta-\alpha)\big)\\ &= \begin{cases} 1 & \text{if } 2p|(\beta-\alpha),\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{13}$$

In other words, $\xi_{\alpha,\beta} \neq 0$ if and only if $\alpha \equiv \beta \pmod{2p}$. Therefore property (iii) of Lemma 3.1 won't give any dependency relation among the $h_\alpha(\tau)$. Now let χ be an odd character. In this case, as observed before we have $h_0(\tau) = 0 = h_p(\tau)$. Also, by the same property (i) of Lemma 3.1, we have $h_\alpha(\tau) = -h_{-\alpha}(\tau)$. So, in the remaining $2p-2$ components, if we assume that $p-1$ of them are zero, then $\phi(\tau, z) = 0$. This completes the first part. If χ is an even character, we cannot conclude that h_0 and h_p are zero. Therefore, combining with the property $h_\alpha(\tau) = h_{-\alpha}(\tau)$ we see that one needs $p - 1 + 2 = p + 1$ components h_α to determine the given Jacobi form ϕ.

Remark 3.4. The number of components as obtained in Theorem 2.3 is minimal in the sense that it is not possible to reduce the number by using the properties stated in Lemma 3.1. It is unclear whether one could obtain smaller bounds using other methods.

3.3. **Proof of Theorem 2.4.** As $m = pq$, p, q are distinct odd primes with $\gcd(pq, N) = 1$, by Lemma 3.2, we have

$$\xi_{\alpha,-\beta} = \frac{1}{\sqrt{pq}} \left(\frac{-N}{pq}\right) \epsilon_{pq}\, e_{4pq}\left(N'(\alpha+\beta)^2\right),$$

if $2|(\alpha+\beta)$, where N' is an integer such that $(4N)N' \equiv 1 \pmod{pq}$. We group the residue classes modulo $2pq$, which are square roots of the squares modulo $4pq$, as follows.

(a) those having 4 square roots with $\gcd(\mu, pq) = 1$, which are $\phi(pq)/2$ in number;

(b) those having 2 square roots with $\gcd(\mu, pq) = p$ or q, which are $(\phi(p) + \phi(q))$ in number;

(c) those having only one square root, which are 2 in number, namely $\{0, pq\}$.

Now assume that $h_\alpha = 0$ for some α with $2|\alpha$ and $\gcd(\alpha, pq) = 1$. We claim that $h_\beta = 0$ for all $\beta \pmod{2pq}$, where β is even. We already know that $h_\alpha = 0$, implies $h_{-\alpha} = 0$. Note that ν and $-\nu$ satisfy the congruence $\nu^2 \equiv \alpha^2 \pmod{4pq}$. Now applying the inversion formula (iii) of Lemma 3.1 we get

$$\xi_{\alpha,\nu} h_\nu(\tau) + \xi_{\alpha,-\nu} h_{-\nu}(\tau) = 0,$$
$$\xi_{-\alpha,\nu} h_\nu(\tau) + \xi_{-\alpha,-\nu} h_{-\nu}(\tau) = 0,$$

or

$$h_\nu(\tau) \;=\; -\frac{\xi_{\alpha,-\nu}}{\xi_{\alpha,\nu}}\, h_{-\nu}(\tau) \;=\; -\frac{\xi_{-\alpha,-\nu}}{\xi_{-\alpha,\nu}}\, h_{-\nu}(\tau).$$

Claim 2.

$$\frac{\xi_{\alpha,-\nu}}{\xi_{\alpha,\nu}} \neq \frac{\xi_{-\alpha,-\nu}}{\xi_{-\alpha,\nu}} \tag{14}$$

equivalently

$$\xi_{\alpha,-\nu}\, \xi_{-\alpha,\nu} \neq \xi_{\alpha,\nu}\, \xi_{-\alpha,-\nu}$$

Proof of Claim 2. Assume that the claim is not true. Then by using (8), we get

$$\xi^2_{\alpha,-\nu} = \xi^2_{\alpha,\nu} \quad \text{or} \quad \xi_{\alpha,-\nu} = \pm\, \xi_{\alpha,\nu}.$$

As in the proof of Theorem 2.2, we have

$$\xi_{\alpha,-\nu} = \pm\, \xi_{\alpha,\nu}. \quad \iff \quad e_{pq}\left(-N'\alpha\nu\right) = \pm 1,$$

Since α, ν are relatively prime to pq, the above relation is not true, which proves the claim. This shows that

$$h_{-\nu}(\tau) = 0 = h_\nu(\tau).$$

Next, by assuming that $h_\alpha = 0$, $2|\alpha$ with $\gcd(\alpha, pq) = 1$, we prove that $h_\beta = 0$ for all $2|\beta$, $\gcd(\beta, pq) = 1$ such that $\beta^2 \not\equiv \alpha^2 \pmod{4pq}$. Observe that the above arguments show that $h_\nu = 0$ for all ν in the same square class α^2 modulo $4pq$ (which are 4 in number, namely α, $-\alpha$, ν and $-\nu$). Let a, b be residue classes modulo $2pq$ such that $2|a$, $2|b$, $a^2 \equiv b^2 \pmod{4pq}$ and $a^2 \not\equiv \alpha^2 \pmod{4pq}$. Now applying the inversion formula (iii) of Lemma 3.1 for $h_\alpha, h_{-\alpha}, h_\nu, h_{-\nu}$, we have the following.

$$\begin{pmatrix} \xi_{\alpha,a} & \xi_{\alpha,-a} & \xi_{\alpha,b} & \xi_{\alpha,-b} \\ \xi_{-\alpha,a} & \xi_{-\alpha,-a} & \xi_{-\alpha,b} & \xi_{-\alpha,-b} \\ \xi_{\nu,a} & \xi_{\nu,-a} & \xi_{\nu,b} & \xi_{\nu,-b} \\ \xi_{-\nu,a} & \xi_{-\nu,-a} & \xi_{-\nu,b} & \xi_{-\nu,-b} \end{pmatrix} \begin{pmatrix} h_a(\tau) \\ h_{-a}(\tau) \\ h_b(\tau) \\ h_{-b}(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let Λ denote the above 4×4 matrix. If $\det \Lambda \neq 0$, then it follows that

$$h_a(\tau) = h_{-a}(\tau) = h_b(\tau) = h_{-b}(\tau) = 0.$$

Thus, assuming that $h_\alpha = 0$ for some α with $2|\alpha$, $\gcd(\alpha, pq) = 1$, we have shown that h_β is zero for all other β even such the $\gcd(\beta, pq) = 1$. Therefore, it remains to prove in this case that $\det \Lambda \neq 0$. This is equivalent to show that det $\Lambda' \neq 0$, where $\Lambda' =$

$$\begin{pmatrix} e_{4pq}(-N'(\alpha-a)^2) & e_{4pq}(-N'(\alpha+a)^2) & e_{4pq}(-N'(\alpha-b)^2) & e_{4pq}(-N'(\alpha+b)^2) \\ e_{4pq}(-N'(\alpha+a)^2) & e_{4pq}(-N'(\alpha-a)^2) & e_{4pq}(-N'(\alpha+b)^2) & e_{4pq}(-N'(\alpha-b)^2) \\ e_{4pq}(-N'(\nu-a)^2) & e_{4pq}(-N'(\nu+a)^2) & e_{4pq}(-N'(\nu-b)^2) & e_{4pq}(-N'(\nu+b)^2) \\ e_{4pq}(-N'(\nu+a)^2) & e_{4pq}(-N'(\nu-a)^2) & e_{4pq}(-N'(\nu+b)^2) & e_{4pq}(-N'(\nu-b)^2) \end{pmatrix}.$$

Write

$$\Lambda' = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are 2×2 matrices. It is easy to see that $AC = CA$. Also A is non-singular, as $\det A = 0$ implies that $e_{4pq}(4N'\alpha a) = e_{4pq}(-4N'\alpha a)$ or equivalently $-2N'\alpha a \equiv 0 \pmod{pq}$, a contradiction since $\gcd(a, pq) = 1 = \gcd(\alpha, pq)$.
Now $\det \Lambda' = R^2 - S^2$, where

$$R = e_{2pq}(N'(\alpha a+\nu b)) + e_{2pq}(-N'(\alpha a+\nu b)) - e_{2pq}(N'(\nu a+\alpha b)) - e_{2pq}(-N'(\nu a+\alpha b))$$

and

$$S = e_{2pq}(N'(\alpha a-\nu b)) + e_{2pq}(-N'(\alpha a-\nu b)) - e_{2pq}(N'(\nu a-\alpha b)) - e_{2pq}(-N'(\nu a-\alpha b)).$$

It can be seen that $\det \Lambda' = 0$ implies that $R = \pm S$, which is impossible. This completes the proof of the even case. Similarly, assuming $h_\beta = 0$ for some β odd with $\gcd(\beta, pq) = 1$, we can show that $h_\alpha = 0$ for all α odd, $\gcd(\alpha, pq) = 1$.

Finally, assume that $h_\alpha = 0$ for some α with $2|\alpha$ and $\gcd(\alpha, pq) = 1$,. We claim that $h_\beta = 0$ for all $\beta \pmod{2pq}$, where β is even and $\gcd(\beta, pq) = p$ or q. Suppose that $\gcd(\beta, pq) = p$ i.e $\beta = p\beta'$ where $\gcd(\beta', q) = 1$. We already know that $h_\alpha = 0$, implies $h_{-\alpha} = 0$. Now applying the inversion formula (iii) of Lemma 3.1, we get

$$\xi_{\alpha,\beta} h_\beta(\tau) + \xi_{\alpha,-\beta} h_{-\beta}(\tau) = 0.$$

$$\xi_{-\alpha,\beta} h_\beta(\tau) + \xi_{-\alpha,-\beta} h_{-\beta}(\tau) = 0.$$

or

$$h_\beta(\tau) \;=\; -\frac{\xi_{\alpha,-\beta}}{\xi_{\alpha,\beta}}\, h_{-\nu}(\tau) \;=\; -\frac{\xi_{-\alpha,-\beta}}{\xi_{-\alpha,\beta}}\, h_{-\beta}(\tau).$$

But by Claim 1 we see that $\xi_{\alpha,-\beta} \neq \pm \xi_{\alpha,\beta}$, which implies $h_\beta = 0 = h_{-\beta}$.

Since the cases β even and $\gcd(\beta, pq) = q$ and β odd and $\gcd(\beta, pq) = p$ or q are similar, this completes the proof.

3.4. **Proof of Theorem 2.5.** In this case we group the residue classes (modulo $2p^2$) which are square roots of the squares modulo $4p^2$ as follows.

(a) The square roots of a^2, $\gcd(a,p) = 1$ are $\pm a$. The total number of such squares are $\phi(p^2) = p(p-1)$ and the total number of square roots are $2p(p-1)$.
(b) The square roots of 0 are given by $2ip$, where $i = 0, 1, 2, \ldots, p-1$. The total number of these square roots are p.
(c) The square roots of p^2 are given by $2(i+1)p$, where $i = 0, 1, 2, \ldots, p-1$. The total number of these square roots are p.

By Lemma 3.2, the value of the Gauss sum in this case equals

$$\xi_{\alpha,\beta} = \frac{1}{p}\, e_{p^2}\left((4N)^{-1}\left(\frac{\beta-\alpha}{2}\right)^2\right),$$

where $(4N)^{-1}4N \equiv 1 \pmod{p^2}$, if $2|(\beta-\alpha)$. Now we assume that $h_a(\tau) = 0$ for some a with $2|a$ and $\gcd(a,p) = 1$. We show that $h_b = 0$ for all b even with $\gcd(b,p) = 1$. We already know that $h_a = 0$ implies $h_{-a} = 0$. Then as before, applying the inversion formula (iii) of Lemma 3.1, we get

$$\xi_{a,b}h_b(\tau) + \xi_{a,-b}h_{-b}(\tau) = 0,$$
$$\xi_{-a,b}h_b(\tau) + \xi_{-a,-b}h_{-b}(\tau) = 0$$

or

$$h_b(\tau) \;=\; -\frac{\xi_{a,-b}}{\xi_{a,b}}\, h_{-b}(\tau) \;=\; -\frac{\xi_{-a,-b}}{\xi_{-a,b}}\, h_{-b}(\tau).$$

Now proceeding as in the proof of Theorem 2.2, we see that the above relation does not hold, since $\gcd(p, 2Nab) = 1$. Now using $h_a = \chi(-1)h_{-a}$, among the p components $h_{2ip}(\tau), 0 \le i \le p-1$, $\frac{p-1}{2}+1$ of them determine the rest. Similarly, among the p components $h_{(2i+1)p}(\tau)$, $0 \le i \le p-1$, $\frac{p-1}{2}+1$ of them determine the rest. Note that if χ is an odd character, then $h_0 = 0 = h_{p^2}$. The proof is now complete.

Remark 3.5. We feel that the components h_a, $2|a$ and h_b, $2 \nmid b$, where $\gcd(ab,p) = 1$ will determine the Jacobi form $\phi(\tau, z)$ in Theorem 2.5. We have already shown that these components determine the other components h_μ such that $\gcd(\mu, p) = 1$. So, it remains to prove that one of h_a, $2|a$ and one of h_b, $2 \nmid b$ with $\gcd(ab,p) = 1$ will determine the other h_μ corresponding to the square classes 0 and p^2. Assuming that $h_a = 0 = h_b$, $2|a$, $2 \nmid b$ with $\gcd(ab,p) = 1$, we use property (iii) of Lemma 3.1 to get a system of equations involving h_μ corresponding to the square classes 0 or p^2. Using the fact that $h_\mu = \chi(-1)h_{-\mu}$, we obtain a system of equations in $(p+1)/2$ variables and the number of such equations will be $(p^2-1)/2 - [(p-1)/2]$, where $[x]$ denotes the greatest integer $\le x$. So, it is enough to show that the resulting matrix has rank $(p+1)/2$. We have verified this fact for the first few primes. We do not know whether this fact can be proved in generality.

Remark 3.6. The result for the case of integral weight Jacobi forms obtained by Skogman [4, Theorem 3.5] and the corresponding result for the case of half-integral weight Jacobi forms obtained in Theorem 2.5 give different characterizations when $m = p^2$. This is mainly due the appearance of quadratic Gauss sum in property (iii) of Lemma 3.1, whereas exponentials appear in the case of integral weight. Moreover, the appearance of the quadratic Gauss sum makes it difficult to construct various h_μ functions from one another as was done in Skogman's work [4, §5].

Remark 3.7. In principle, one can obtain characterization in the case of square-free index as done by Skogman [5].

Acknowledgment. We thank the referees for their comments and suggestions.

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