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6th Semester Project – Solving 1+1 Dimension Hydrodynamics Equations and Studying its Applications

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Project Guide

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Abstract

This will be written atlast.

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CHAPTER 1

Introduction

HYDRODYNAMICS is concerned with the study of dynamics of fluids(gases and liquids). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. This means that any small volume element in the fluid is always supposedly so large that it still contains a very great number of molecules. Accordingly, when we speak of infinitely small elements of volume, we shall always mean those which are "physically" infinitely small, i.e. very small compared with the volume of the body under consideration, but large compared with the distances between the molecules. The expressions fluid particle and point in a fluid are to be understood in a similar sense. If, for example, we speak of the displacement of some fluid particle, we mean not the displacement of an individual molecule, but that of a volume element containing many molecules, though still regarded as a point.The necessity of allowing for relativistic effects in fluid dynamics may not be only due to large velocity of the macroscopic flow (comparable with the speed of light).The equations of fluid dynamics are considerably modified also when this velocity is large but those of the microscopic motion of the fluid particles are large.

Relativistic Hydrodynamics Equations are used as model to study Relativistic Heavy Ion Collisions at various Heavy Ion Colliders. The domain upto which relativistic hydrodynamics is applicable in Heavy Collisions is restricted to only to systems under **local equilibrium**. These particles must be interacting with each other to reach equilibrium.**Local thermodynamic equilibrium (LTE) means that the intensive parameters(scale invariant quantities e.g.pressure density) are varying in space and time, but are varying so**

slowly that, for any point, one can assume thermodynamic equilibrium in some neighborhood about that point. Conditions for Thermodynamics equilibrium

- For a completely isolated system,entropy(S) is maximum at thermodynamic equilibrium.
- For a system with controlled constant temperature and volume, Helmholtz free energy(A) is minimum at thermodynamic equilibrium.
- For a system with controlled constant temperature and pressure, Gibbs free energy(G) is minimum at thermodynamic equilibrium.

CHAPTER 2

Domain in which hydrodynamics is applicable

RELATIVISTIC Heavy ion particles and their collisions can be modelled with hydrodynamics only if their interactions are frequent enough for establishing an equilibrium(local). In order to quantify this frequency of collisions, one way is to choose the mean free path λ, which is the average distance a particle covers before colliding for the 2nd time, as very very less than the total size of the system say, L. Thus, by setting $\lambda \ll L$, we can apply hydrodynamics formulation. Now we derive the expression for mean free path as function of temperature. In kinetic gas theory atoms are considered as hard spheres, i.e. they only interact if their distance becomes smaller than the sum of their gas-kinetic radii.[1] Therefore the area around an atom(with radius r_1) in which another atom(with radius r_2) will be scattered, defined as the cross section σ, is given by

$$\sigma = \pi(r_1 + r_2)^2 \tag{2.1}$$

For an ensemble of such atoms, only interacting by collisions, the ideal gas law

$$P = nk_BT \tag{2.2}$$

holds with n being the particle density where k_B is Boltzmann constant and T is temperature. Now, lets say, in an ensemble of N atoms ,there are dN atoms getting scattered by an distance dx then

$$\frac{dN}{dx} = Nn\sigma$$

[1]The distance from nucleus(considered as point particle) at which the effect of the deformation of the wavefunctions by Van der Waals interactions is almost negligible.

In a beam of N_0 atoms flying through a gas a number

$$N = N_0 e^{n\sigma x} \tag{2.3}$$

will be scattered along a distance x. By definition and using above equation one finds for the mean free path

$$\lambda = \frac{1}{N_0}\int_0^\infty x \cdot |\frac{dN}{dx}|dx = n\sigma \cdot \int_0^\infty xe^{-n\sigma x}dx = \frac{1}{n\sigma} \tag{2.4}$$

Thus, now by applying our limit on mean free path we can approximate any such system by the relativistic hydrodynamics model.

Now, in our system of relativistic particles, if we have,

- $\lambda > L$(Size of System)

  Consequence:- The particle leaves the system before interacting with other particles in the system. Examples are photons inside a system of particles which are interacting strongly.

- $\lambda \approx L$

  Consequence:- The interaction can be modelled using multiple collision model. It doesn't reach local thermal equilibrium

- $\lambda \ll L$

  Consequence:- In this limit, successive collision model fails and then we need to treat them as gas of particles and then it can attain local thermal equilibrium and hydrodynamics scheme can be used.

CHAPTER 3

Relativistic Fluid Dynamics

Now, for studying relativistic hydrodynamics Equation of State(EOS),which a continuous system, a simplistic classical approach is taken, where we start with a classical model with particles attached with springs, and taking a infinite number of such particles joined by springs, then the equilibrium length is made to tend to 0. Thus, we will make a transition from a discrete system of particles to a continuum system. So lets see how it works out.

3.1 Mass points connected by springs- A discrete system

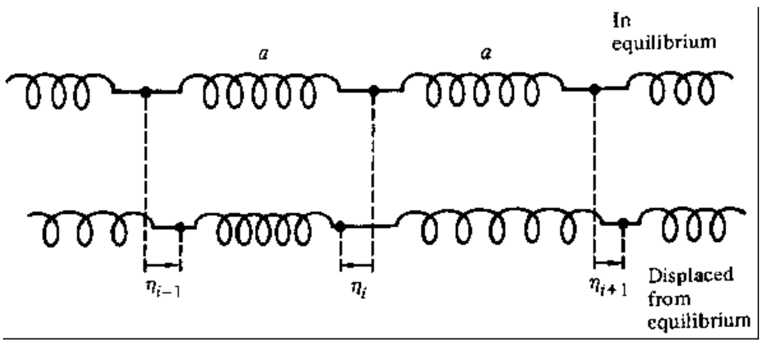

Figure 3.1: Discrete system:Mass points connected by springs

Variables in this system

a = equilibrium length of spring.

η_i = displacement of the i^{th} particle from its equilibrium position.

m = mass point,

Starting with the Kinetic Energy expression,

$$T = \frac{1}{2}\sum_i m\dot{\eta_i}^2 \tag{3.1}$$

The corresponding potential energy is the sum of the potential energies of each spring as the result of being stretched or compressed from its equilibrium length:

$$V = \frac{1}{2}\sum_i k(\eta_{i+1} - \eta_i)^2 \tag{3.2}$$

Combining 3.1 and 3.2, the Lagrangian for the system is,

$$L = T - V = \frac{1}{2}\sum_i [m\dot{\eta_i}^2 - k(\eta_{i+1} - \eta_i)^2] \tag{3.3}$$

which can also be written as

$$L = \frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{\eta_i}^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right] = \sum_i aL_i \tag{3.4}$$

This particular form is chosen for convenience in going to the limit of a continuous rod as a approaches 0. On taking limit as $a \to 0$:

- $\frac{m}{a} = \mu$
- $ka = Y$(Young's Modulus)[1]

The above limit is hard to see, unless we consider Hooke's Law:

$$F = ka(\frac{\eta_{i+1} - \eta_i}{a}) \tag{3.5}$$

and,

$$\text{Force(or Pressure)}^2 = Y\xi \tag{3.6}$$

Thus, as $a \to 0$, in ka, $k \to \infty$. Since, physically we can also visualize that as we contract the length of our spring to a single point, the spring becomes nearly impossible to compress, thus increasing the spring constant to ∞.

[1]Please don't check the dimensions now, since we are working with one dimension the concept of Pressure is not so well-defined, Force and Pressure are used interchangeably,

[2]Working in One Dimension makes us to use pressure and force interchangeably

3.2 Transition from Discrete to Continuous System

As our aim was to go to continuous system, the index i's need to be replaced, in other words, the η_i's are replaced by their continuous counterparts as $\eta(x)$. So accordingly,

$$\lim_{a\to 0}\frac{\eta_{i+1}-\eta_i}{a}=\lim_{a\to 0}\frac{\eta(x+a)-\eta(x)}{a}=\frac{d\eta}{dx}$$

Thus, finally our Lagrangian in eq(3.3) looks like this for our continuous system(note the summation changes to a integral),

$$L=\frac{1}{2}\int\left[\mu\dot{\eta}^2-Y\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^2\right]\mathrm{d}x \tag{3.7}$$

This analogy which we used for 1-dimension, can be extended to 3-dimension, then we define the quantity inside the integral as Lagrangian Density($\mathcal{L}$),

$$L=\int\int\int\mathcal{L}\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{3.8}$$

3.3 Change of gears:Use of compact notations

Compact notations[3], can simplify our life, a bit. The generalised coordinate we are working with is $\eta(x)$, and since we are working with Lagrangian density instead of the Lagrangian,our **Euler Lagrange Equation** changes accordingly.[4]

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_x}{dt}\right)}\right)+\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_x}{dx}\right)}\right)+\frac{d}{dy}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_x}{dy}\right)}\right)+\frac{d}{dz}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_x}{dz}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\eta_x}=0\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_y}{dt}\right)}\right)+\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_y}{dx}\right)}\right)+\frac{d}{dy}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_y}{dy}\right)}\right)+\frac{d}{dz}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_y}{dz}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\eta_y}=0\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_z}{dt}\right)}\right)+\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_z}{dx}\right)}\right)+\frac{d}{dy}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_z}{dy}\right)}\right)+\frac{d}{dz}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{d\eta_z}{dz}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\eta_z}=0
\end{aligned} \tag{3.9}$$

All this equations can written in compact notations as,

$$\frac{d}{dx^\nu}\left(\frac{\partial\mathcal{L}}{\partial\eta_{\rho,\nu}}\right)-\frac{\partial\mathcal{L}}{\partial\eta_\rho}=0 \tag{3.10}$$

Thus, it can be seen that the Lagrangian density($\mathcal{L}$) is a function of only $\mathcal{L}(\eta,\frac{d\eta}{dx},\frac{d\eta}{dt},x,y,z,t)$
Now for 1-D case, the equation of motion turns out to be,

$$\mu\frac{d^2\eta}{dt^2}-Y\frac{d^2\eta}{dx^2}=0 \tag{3.11}$$

[3]Use of tensorial notations, though the name sounds gaudy. Relativity demands they be put in relativistically invariant forms, writing in Tensors ensures that.

[4]The complete derivation can be found in Classical Mechanics, Herbert Goldstein pg-565

3.4 Energy Momentum Tensor

Till now, using Euler Lagrange equations we can find the Equations of Motion. If our Lagrangian is independent of one of the cyclic generalised coordinates, then the conjugate generalised momenta is conserved. So in search of that constant of motion, we proceed to write

$$\frac{d\mathcal{L}}{dx^\mu} = \frac{\partial \mathcal{L}}{\partial \eta_\rho}\eta_{\rho,\mu} + \frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}}\eta_{\rho,\mu\nu} + \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{3.12}$$

By the Euler Lagrange Equation (3.10),

$$\frac{d}{dx^\nu}\left[\frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}}\eta_{\rho,\mu} - \mathcal{L}\delta^\nu_\mu\right] = -\frac{\partial \mathcal{L}}{\partial x^\mu} \tag{3.13}$$

Now, here we define the **Canonical Energy Momentum Tensor**. So here if *RHS is 0, then the conjugate momenta corresponding to x^μ is constant of motion.*

$$T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial \eta_{\rho,\nu}}\eta_{\rho,\mu} - \mathcal{L}\delta^\nu_\mu \tag{3.14}$$

3.5 Components of Energy Momentum Tensor

3.5.1 T^0_0 component

$$T^0_0 = \frac{\partial \mathcal{L}}{\partial \dot{\eta}_\rho}\dot{\eta}_\rho - \mathcal{L} \tag{3.15}$$

From the definition of canonical momentum as,

$$p_\rho = \frac{\partial \mathcal{L}}{\partial \dot{\eta}_\rho} \tag{3.16}$$

It turns out that

$$\boxed{T^0_0 = p_\rho \dot{\eta}_\rho - \mathcal{L} = \epsilon(\text{Energy Density of system})}$$

3.5.2 T^i_0 component

In continuum limit the Lagrangian density was written,(in 1 dimension)(3.7)

$$\mathcal{L} = \mu\dot{\eta}^2 - Y\left(\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)^2 \tag{3.17}$$

So, for calculating the T^i_0th component we need,

$$\frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dx}} = -Y\frac{d\eta}{dx} \tag{3.18}$$

Thus computing T^1_0's(for 1 dimension)

$$T^1_0 = -Y\frac{d\eta}{dx}\dot{\eta}$$

where $-Y\frac{d\eta}{dx}$ =Force(or Stress in 1 dimension), Therefore,

T_0^i =Rate of energy being transferred per unit time(energy current)

3.5.3 T_i^0 component

Similarly, for calculating T_i^0th component we need,

$$\frac{\partial \mathcal{L}}{\partial \frac{d\eta}{dt}} = \mu \frac{d\eta}{dt} \tag{3.19}$$

Momentum density(in 1-D momentum per unit length) can be given by $\mu\dot{\eta}$, but when wave motion(from 3.11) takes place, a net mass change in the length dx at any time is given by,

$$\mu[\eta(x) - \eta(x+dx)] = -\mu\frac{d\eta}{dx}dx$$

This implies,
Net Momentum= $-\mu\dot{\eta}\frac{d\eta}{dx}dx$

$T_i^0 = -\mu\dot{\eta}\frac{d\eta}{dx}$(Momentum Density)

3.5.4 T_i^j component

For interpreting the T_i^j components, we take a slightly different approach, where we assume $\mathcal{L}$ represents a free field($\mathcal{L}$ is independent of x^μ). This implies $T^\nu_{\mu,\nu} = 0$, which upon expansion,

$$T^\nu_{\mu,\nu} = \frac{dT^0_\mu}{cdt} + \frac{dT^i_\mu}{dx^i} = 0 \tag{3.20}$$

Now, integrating both sides over a volume $\mathcal{V}$,

$$\frac{d}{cdt}\int T^0_\mu dV = -\int \nabla \cdot T_\mu dV \tag{3.21}$$

Applying Gauss theorem,

$$\frac{d}{cdt}\int T^0_\mu dV = -\oint T^\nu_\mu dS_\nu \tag{3.22}$$

Here the closed integral is over a closed surface enclosing the volume $\mathcal{V}$. In this equation, we already know T^0_μ represents momentum density. Volume integral of momentum density over Volume $\mathcal{V}$ represents the momentum, and consequently its time derivative give us the **Force**.

$L.H.S$ =Force$\Rightarrow T^\nu_\mu$(only the 3×3 matrix) represents Pressure along ν'th direction.

3.6 Energy momentum tensor in Relativistic Hydrodynamics

Now, that we know all the components of the Energy Momentum Tensor, we construct the Energy Momentum Tensor for Relativistic Heavy Ion Collision in the Local Rest Frame(LRF) of the particles,

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

where

ϵ =Energy Density

P = Pressure in Local Rest Frame of the Particles.

Since we took an **isotropic and perfect fluid**, the Pressure is equally distributed and only diagonal components survive,

$$T^{ii} = \frac{\Delta p_i}{\Delta t \Delta x^j \Delta x^k}$$

where Δp_i is the momentum change along ith direction, and Δx^j is displacement along jth direction.

3.7 Fluid in any general inertial frame

In order to obtain the Energy-Momentum Tensor in any general inertial frame, we give it a Lorentz Boost along arbitrarily chosen $\mathbf{u}^\mu(\gamma, \gamma v_x, \gamma v_y, \gamma v_z)$ direction,

$$\mathbf{u}^\mu = \Lambda^\mu_\nu \mathbf{u}^\nu_R \tag{3.23}$$

where $\mathbf{u}^\nu_R$ is the four velocity unit vector in the rest frame of the system,

$$\mathbf{u}^\nu_R = (1, 0, 0, 0)$$

and Λ^μ_ν is the Lorentz Transformation. In short,

$$\mathbf{u}^\mu = \Lambda^\mu_0 \tag{3.24}$$

The metric tensor used for spatial transformation is also Lorentz boosted,

$$g^{\rho\sigma} = g^{\mu\nu}_R \Lambda^\rho_\mu \Lambda^\sigma_\nu \tag{3.25}$$

Since

$$g_R^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now,

$$g^{\rho\sigma} = g_R^{00}\Lambda_0^\rho\Lambda_0^\sigma + g_R^{ii}\Lambda_i^\rho\Lambda_i^\sigma \tag{3.26}$$

Thus,

$$\Lambda_i^\rho\Lambda_i^\sigma = \mathbf{u}^\rho\mathbf{u}^\sigma - g^{\rho\sigma} \tag{3.27}$$

On transforming the $T_R^{\mu\nu}$,

$$\begin{aligned} T^{\rho\sigma} &= T_R^{\mu\nu}\Lambda_\mu^\rho\Lambda_\nu^\sigma \\ &= \epsilon\Lambda_0^\rho\Lambda_0^\sigma + P\Lambda_i^\rho\Lambda_i^\sigma \\ &= \epsilon\mathbf{u}^\rho\mathbf{u}^\sigma + P(\mathbf{u}^\rho\mathbf{u}^\sigma - g^{\rho\sigma}) \\ &= (\epsilon + P)\mathbf{u}^\rho\mathbf{u}^\sigma - g^{\rho\sigma}P \end{aligned} \tag{3.28}$$

3.8 Hydrodynamics Equations

As in 3.13, we set $RHS = 0$, implies Lagrangian is independent of space(x, y, z) and time(ct), which again implies their generalised conjugate momenta, momentum(p_x, p_y, p_z) and energy(ϵ) are conserved. The Energy and Momentum conservation equation is:

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.29}$$

Entropy current$(\mathbf{J}^\mu = s\mathbf{u}^\mu)$ and baryon number density$(\mathbf{J}^\mu = n_{bar}\mathbf{u}^\mu)$ current conservation equation is given by,

$$\partial_\mu \mathbf{J}^\mu = 0 \tag{3.30}$$

In rest frame, the derivative of energy momentum tensor gives,

$$\partial_0 T^{00} = \frac{\partial \epsilon}{\partial t} = 0 \tag{3.31}$$

$$\partial_i T^{ii} = \frac{\partial P}{\partial x^i} = 0 \tag{3.32}$$

Thus the energy density is forced to be constant with time but could have spatial variations while the pressure is constant over all space (no pressure gradients) but could depend on time. For a moving fluid, more work is needed. To obtain a scalar quantity from the requirement of energy-momentum conservation, we contract 3.29 with the velocity $\mathbf{u}_\nu$,

$$\mathbf{u}_\nu \partial_\mu T^{\mu\nu} = 0 \tag{3.33}$$

$$\begin{aligned} 0 &= \mathbf{u}_\nu \partial_\mu T^{\mu\nu} \\ &= \mathbf{u}_\nu \partial_\mu [(\epsilon + P)\mathbf{u}^\rho \mathbf{u}^\sigma - g^{\rho\sigma} P] \end{aligned} \tag{3.34}$$

Upon expanding,

$$\begin{aligned} 0 = \mathbf{u}_\nu \mathbf{u}^\mu \mathbf{u}^\nu \partial_\mu(\epsilon + P) + (\epsilon + P)(\mathbf{u}_\nu \mathbf{u}^\mu \partial_\mu \mathbf{u}^\nu \\ \mathbf{u}_\nu \mathbf{u}^\nu \partial_\mu \mathbf{u}^\mu) - \mathbf{u}_\nu \partial^\nu P \end{aligned} \tag{3.35}$$

Since $\mathbf{u}$ is a unit four-vector, $\mathbf{u}_\nu \mathbf{u}_\nu = \mathbf{u}^2 = 1$. In addition, this property also gives us $\mathbf{u}_\nu \partial_\mu \mathbf{u}^\nu = (\frac{1}{2})\partial_\mu(\mathbf{u}_\nu \mathbf{u}^\nu) = 0$. The derivative is zero since $\mathbf{u}^2 = 1$.
Using these expressions we get,

$$\boxed{\mathbf{u}^\nu \partial_\nu \epsilon + (\epsilon + P)\partial_\nu \mathbf{u}^\nu = 0} \tag{3.36}$$

Now from 1st law of thermodynamics,

$$\epsilon + P = Ts + \mu n_{bar} \tag{3.37}$$

where

$\epsilon =$ energy density

$P =$ Pressure

$T =$ Temperature

$s =$ Entropy density

$u =$ Chemical Potential

$n_{bar} =$ Baryonic Density

Now keeping T,P and μ fixed we can write $\partial_\nu \epsilon = T\partial_\nu s + \mu\partial_\nu n_{bar}$, Replacing directly into 3.36,

$$\begin{aligned} 0 &= T\mathbf{u}^\nu \partial_\nu s + \mu \mathbf{u}^\nu \partial_\nu n_{bar} + (Ts + \mu n_{bar})\partial_\nu \mathbf{u}^\nu \\ &= T\partial_\nu (s\mathbf{u}^\nu) + \mu \partial_\nu (n_{bar}\mathbf{u}^\nu). \end{aligned} \tag{3.38}$$

Since the particle current J^ν is just $n_{bar}\mathbf{u}^\nu$, the second term in 3.38 is zero by baryon number conservation. Thus we have,

$$T\partial_\nu (s\mathbf{u}^\nu) = 0 \tag{3.39}$$

One more contraction of 3.29 will result in second hydrodynamics equation. Let us choose a tensor combination $g_{\nu\lambda} - \mathbf{u}_\nu \mathbf{u}_\lambda$, which is perpendicular to u_ν.[5]

$$(g_{\nu\lambda} - \mathbf{u}_\nu \mathbf{u}_\lambda)\partial_\mu [(\epsilon + P)\mathbf{u}^\rho \mathbf{u}^\sigma - g^{\rho\sigma} P] = 0 \tag{3.40}$$

On expansion,

$$\begin{aligned} 0 = \mathbf{u}^\mu \mathbf{u}_\lambda \partial_\mu (\epsilon + P) + (\epsilon + P)(\mathbf{u}^\mu \partial_\mu \mathbf{u}_\lambda + \mathbf{u}_\lambda \partial_\mu \mathbf{u}^\mu) - \partial_\lambda P \\ - (\epsilon + P)(\mathbf{u}_\lambda \mathbf{u}^\mu \mathbf{u}_\nu \partial_\mu \mathbf{u}^\nu + \mathbf{u}_\lambda (\mathbf{u}_\nu \mathbf{u}^\nu)\partial_\mu \mathbf{u}^\mu) \\ - \mathbf{u}_\lambda \mathbf{u}^\mu (\mathbf{u}_\nu \mathbf{u}^\nu)\partial_\mu (\epsilon + P) + \mathbf{u}_\lambda \mathbf{u}^\mu \partial^\nu P \end{aligned} \tag{3.41}$$

Again using $\mathbf{u}_\nu \mathbf{u}^\nu = 1$ and $\mathbf{u}_\nu \partial_\mu \mathbf{u}^\nu = 0$, the equation which survives is

$$\boxed{(\epsilon + P)u^\mu \partial_\mu u_\lambda - \partial_\lambda P + u_\mu u_\lambda \partial^\mu P = 0} \tag{3.42}$$

Again using 3.37, now we keep ϵ, s and n_{nar} are fixed then,

$$\partial_\lambda P = s\partial_\lambda T + n_{bar}\partial_\lambda \mu \tag{3.43}$$

Replacing in 3.42

$$s(\mathbf{u}^\mu \partial_\mu (\mathbf{u}_\lambda T) - \partial_\lambda T) + n_{bar}(\mathbf{u}^\mu \partial_\mu (\mathbf{u}_\lambda \mu) - \partial_\lambda \mu) = 0 \tag{3.44}$$

[5] $\mathbf{u}_\nu g_{\nu\lambda} - \mathbf{u}_\nu \mathbf{u}_\lambda = \mathbf{u}_\lambda - \mathbf{u}_\lambda = 0$

If the net baryon density is zero, the equation becomes easier to handle and upon more simplification and dividing by T gives,

$$0 = \mathbf{u}^{\mu}\partial_{\mu}\mathbf{u}_{\lambda} - \partial_{\lambda}\ln T + \mathbf{u}^{\mu}\mathbf{u}_{\lambda}\partial_{\mu}\ln T \tag{3.45}$$

The entropy current conservation equation can also be rewritten in terms of temperature using $\frac{d\ln T}{d\ln S} = c_s^2$,

$$\partial_{\mu}\mathbf{u}^{\mu} + \frac{1}{c_s^2}\mathbf{u}^{\mu}\partial_{\mu}\ln T \tag{3.46}$$

3.9 Application to Heavy Ion Collisions

In centre of mass frame, the accelerated motion of the nuclei before collision is completely longitudinal.

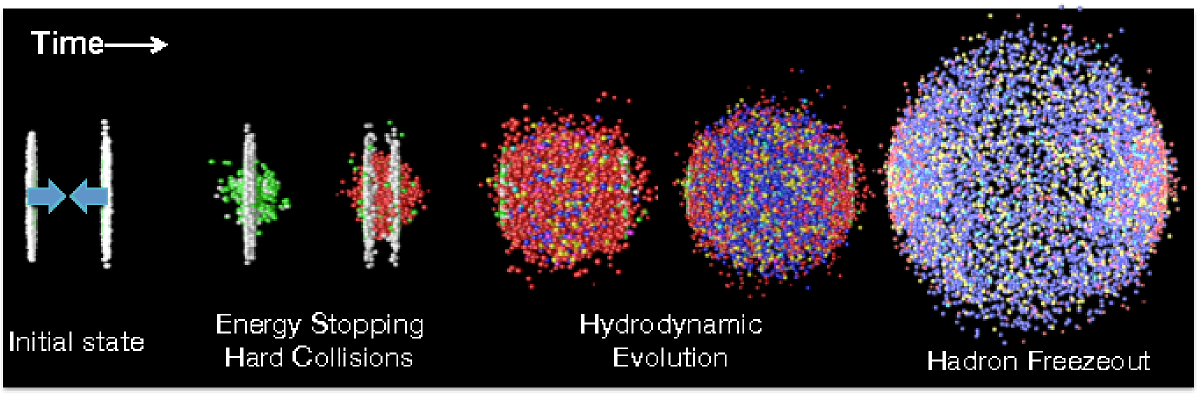

Figure 3.2: Time Evolution of Relativistic Heavy Ion Collision

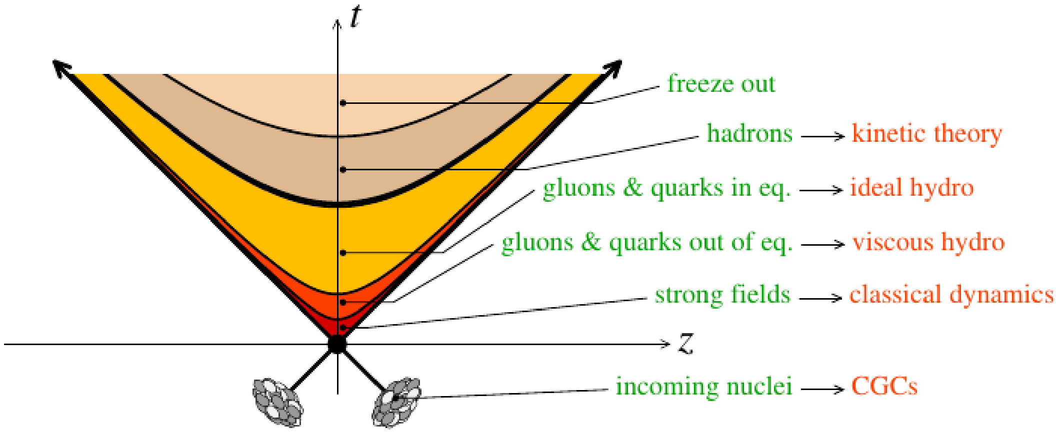

Figure 3.3: Schematic representation of the various stages of a HIC as a function of time t and the longitudinal coordinate z (the collision axis).The light cones represent the velocity of the two colliding nuclei. The 'time' variable which is used in the discussion in the text is the *proper time* $\tau = \sqrt{t^2 - z^2}$, which has a Lorentz invariant meaning and is constant along the hyperbolic curves separating various stages in this figure.

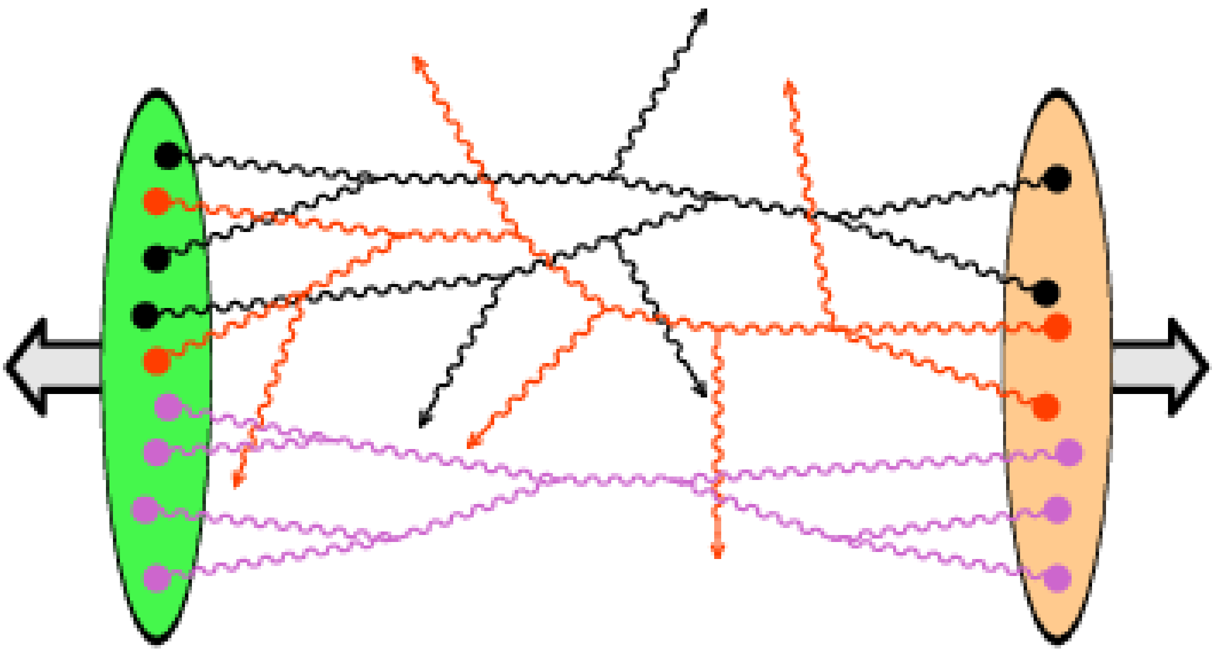

Figure 3.4: Dominant Longitudinal Evolution after HIC

Since longitudinal evolution is dominant, so it is taken as the z coordinate. Some radial component is expected, as we are dealing in cylindrical coordinate system. But due to rotationally symmetry azimuthal angular dependence is ignored. In cylindrical coordinate the $\mathbf{u}^\mu$ can be written as

$$\mathbf{u}^\mu = \frac{1}{\sqrt{1-(v_r^2+v_z^2+v_\phi^2)}}(1, v_z, v_r, v_\phi)$$

Since $v_\phi = 0$,

$$\mathbf{u}^\mu = \frac{1}{\sqrt{1-(v_r^2+v_z^2)}}(1, v_z, v_r)$$

where v_r and v_z are radial and longitudinal components respectively. In general $v_r < v_z$, but here for simplicity we assume $v_r \ll v_z$, then in terms of rapidity variable "θ" this can be written as,(since $v_z = \tanh\theta$)

$$\mathbf{u}^\mu = (\cosh\theta, \sinh\theta, 0, 0)$$

where θ is the fluid velocity. Now lets try to untangle from the compact notations for the equations (3.42), for $\lambda = 0$,

$$\begin{aligned} 0 &= -\partial_0 P + \mathbf{u}_\mu u_0 \partial^\mu P + (\epsilon+P)\mathbf{u}^\mu \partial_\mu u_0 \\ &= -\frac{\partial P}{\partial t} + \cosh\theta\left(\cosh\theta\frac{\partial P}{\partial t} + \sinh\theta\frac{\partial P}{\partial z}\right) \\ &+ (\epsilon+P)\left(\cosh\theta\frac{\partial\cosh\theta}{\partial t} + \sinh\theta\frac{\partial\cosh\theta}{\partial z}\right) \\ &= (\cosh^2\theta - 1)\frac{\partial P}{\partial t} + \cosh\theta\sinh\theta\frac{\partial P}{\partial z} \\ &+ (\epsilon+P)\left(\cosh\theta\sinh\theta\frac{\partial\theta}{\partial t} + \sinh^2\theta\frac{\partial\theta}{\partial z}\right) \end{aligned} \tag{3.47}$$

where we have used $u_0 = \cosh\theta$. Using the identity $\cosh^2\theta - 1 = \sinh^2\theta$ and dividing by $\sinh\theta$, we are left with

$$\begin{aligned} 0 &= \sinh\theta\frac{\partial P}{\partial t} + \cosh\theta\frac{\partial P}{\partial z} \\ &+(\epsilon+P)\left(\cosh\theta\frac{\partial\theta}{\partial t} + \sinh\theta\frac{\partial\theta}{\partial z}\right) \end{aligned} \tag{3.48}$$

for $\lambda = 1$,

$$\begin{aligned} 0 &= -\partial_1 P + \mathbf{u}_\mu u_1 \partial^\mu P + (\epsilon+P)\mathbf{u}^\mu \partial_\mu u_1 \\ &= \tfrac{\partial P}{\partial z} + \sinh\theta\left(\cosh\theta\tfrac{\partial P}{\partial t} + \sinh\theta\tfrac{\partial P}{\partial z}\right) \\ &+(\epsilon+P)\left(\cosh\theta\tfrac{\partial\sinh\theta}{\partial t} + \sinh\theta\tfrac{\partial\sinh\theta}{\partial z}\right) \\ &= (\sinh^2\theta+1)\tfrac{\partial P}{\partial z} + \sinh\theta\cosh\theta\tfrac{\partial P}{\partial t} \\ &+(\epsilon+P)\left(\cosh\theta\sinh\theta\tfrac{\partial\theta}{\partial z} + \cosh^2\theta\tfrac{\partial\theta}{\partial t}\right) \end{aligned}$$

Using $u_1 = \sinh\theta$. Again using similar identities we get back eqn(3.48). Using almost the same routine we can get from 3.36,

$$\begin{aligned} 0 &= \cosh\theta \frac{\partial \epsilon}{\partial t} + \sinh\theta \frac{\partial \theta}{\partial z} \\ &+(\epsilon + P)\left(\sinh\theta \frac{\partial \theta}{\partial t} + \cosh\theta \frac{\partial \theta}{\partial z}\right) \end{aligned} \tag{3.49}$$

Now, let us simplify the equations even more by using light cone variables,space-time rapidity $(\eta_s)^6$ and proper time (τ),

$$\begin{aligned} \eta_s &= \frac{1}{2}\ln\frac{t+z}{t-z} \\ \tau &= \sqrt{t^2 - z^2} \end{aligned} \tag{3.50}$$

Now, t and z can be parametrized as follows(since $\frac{1}{2}\ln\frac{1+x}{1-x} = \tanh^{-1} x \Rightarrow \frac{z}{t} = \tanh\eta_s$),

$$\begin{aligned} t &= \tau\cosh\eta_s \\ z &= \tau\sinh\eta_s \end{aligned} \tag{3.51}$$

Upon transforming the partial derivatives,

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial \eta_s}{\partial t}\frac{\partial}{\partial \eta_s} \\ &= \cosh\eta_s \frac{\partial}{\partial \tau} + \frac{1}{\tau}\sinh\eta_s \frac{\partial}{\partial \eta_s} \end{aligned} \tag{3.52}$$

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial \tau}{\partial z}\frac{\partial}{\partial \tau} + \frac{\partial \eta_s}{\partial z}\frac{\partial}{\partial \eta_s} \\ &= -\sinh\eta_s \frac{\partial}{\partial \tau} + \frac{1}{\tau}\cosh\eta_s \frac{\partial}{\partial \eta_s} \end{aligned} \tag{3.53}$$

Substituting in (3.49) and using trigonometric identities,

$$\begin{aligned} 0 &= \cosh(\theta - \eta_s)\frac{\partial \epsilon}{\partial \tau} + \frac{1}{\tau}\sinh(\theta - \eta_s)\frac{\partial \epsilon}{\partial \eta_s} \\ &+(\epsilon + P)\left(\sinh(\theta - \eta_s)\frac{\partial \theta}{\partial \tau} + \frac{1}{\tau}\cosh(\theta - \eta_s)\frac{\partial \theta}{\partial \eta_s}\right) \end{aligned} \tag{3.54}$$

Finally dividing by $\cosh(\theta - \eta_s)$ and multiplying by τ, we are left with

$$\begin{aligned} 0 &= \tau\frac{\partial \epsilon}{\partial \tau} + \tanh(\theta - \eta_s)\frac{\partial \epsilon}{\partial \eta_s} \\ &+(\epsilon + P)\left(\tau\tanh(\theta - \eta_s)\frac{\partial \theta}{\partial \tau} + \frac{\partial \theta}{\partial \eta_s}\right) \end{aligned} \tag{3.55}$$

Likewise we do the same treatment with (3.48) and we get,

$$\begin{aligned} 0 &= \tau\tanh(\theta - \eta_s)\frac{\partial P}{\partial \tau} + \frac{\partial P}{\partial \eta_s} \\ &+(\epsilon + P)\left(\tau\frac{\partial \theta}{\partial \tau} + \tanh(\theta - \eta_s)\frac{\partial \theta}{\partial \eta_s}\right) \end{aligned} \tag{3.56}$$

[6]This rpidity is different from the rapidity variable θ

Now, we expand the baryon number conservation equation,

$$\partial^{\mu}(n_{bar}\mathbf{u}_{\mu}) = 0 \tag{3.57}$$

$$\begin{aligned} 0 &= \tau\frac{\partial n_{bar}}{\partial\tau} + \tanh\left(\theta - \eta_s\right)\frac{\partial n_{bar}}{\partial\eta_s} \\ &+ n_{bar}\left(\tau\tanh\left(\theta - \eta_s\right)\frac{\partial\theta}{\partial\tau} + \frac{\partial\theta}{\partial\eta_s}\right) \end{aligned} \tag{3.58}$$

Since the entropy current conservation looks the same , so,

$$\begin{aligned} 0 &= \tau\frac{\partial s}{\partial\tau} + \tanh\left(\theta - \eta_s\right)\frac{\partial s}{\partial\eta_s} \\ &+ s\left(\tau\tanh\left(\theta - \eta_s\right)\frac{\partial\theta}{\partial\tau} + \frac{\partial\theta}{\partial\eta_s}\right) \end{aligned} \tag{3.59}$$

3.10 Solution of Hydrodynamics equations

We have three equations(Eq3.55,3.56 and 3.58) with the following variables:

- Energy Density(ϵ)
- Pressure(P)
- Fluid Velocity(θ)
- Baryon Density(n_{bar})

Thus, we need one more equation to describe the complete picture, which would relate ϵ and P. We know the equation of state for a free gas of massless quarks and gluons,

$$\boxed{P = \frac{\epsilon}{3}} \tag{3.60}$$

3.10.1 Initial Conditions

Initial conditions decide the evolution of the fluid further.

Landau Initial Conditions

Landau's model assumed hydrodynamical evolution that begins from rest, complete stopping of the initial nuclei. According to Landau, when two hadrons collide, the collision energy is released into a very small volume in the center of mass. The energy distribution in a small volume can be treated statistically without actually knowing the nuclear interaction. The volume is contracted in the direction of motion, as discussed in the introduction. Landau's final assumption, that the final-state particles are formed instantaneously and immediately leave the collision volume without further interaction, is not justified. First, since hadrons interact strongly, it is unlikely that they would leave the volume without interacting with any of the other particles around them. In addition, particle production is not instantaneous. Finally, the system should expand and the number of produced particles is only 'frozen in' after interactions cease.
Thus according to him,

- $\theta(t = 0, \vec{x}) = 0$

The **Drawback** of this initial condition is that the assumption that the system starts from rest is generally too extreme.

Bjorken Initial Condition

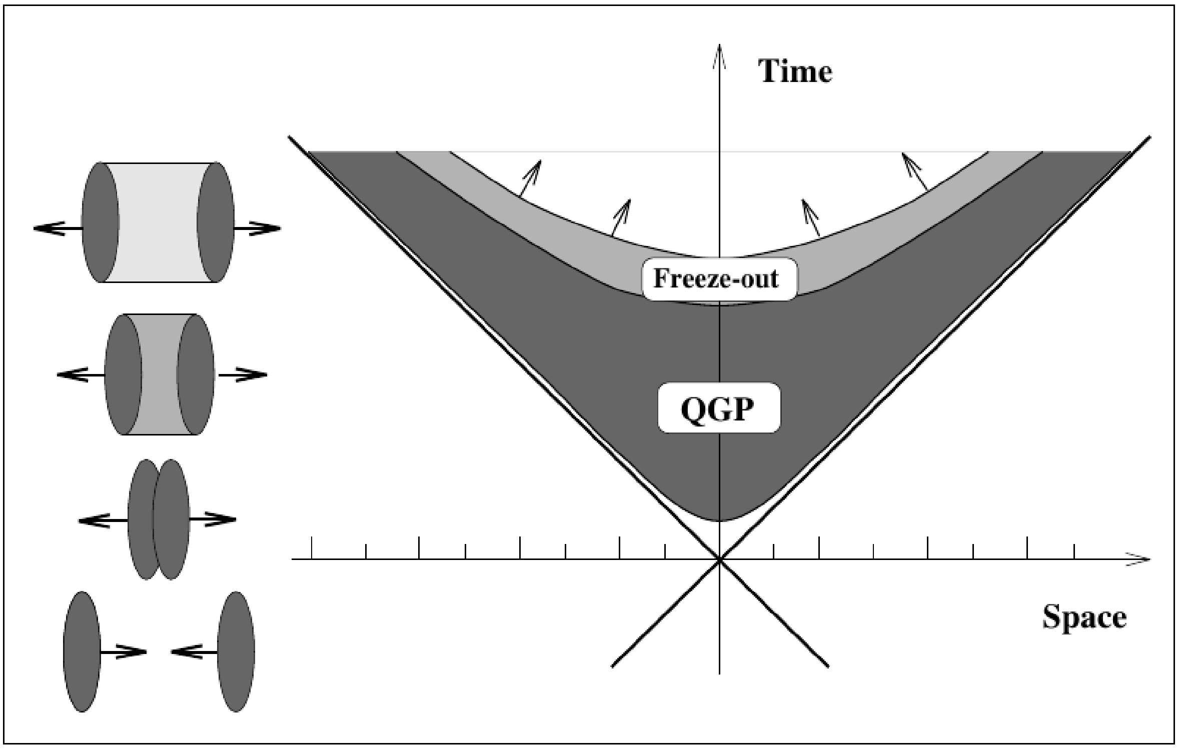

Figure 3.5: A space-time-image illustration of a heavy-ion collision in the ultra- relativistic (Bjorken) collision limit. Left: Lorentz-contracted nuclei collide and separate as a function of the laboratory time (vertical axis). Right: the light cone establishes the causality limit for the flow of energy, which fills the space-time domain between the separating nuclei.

In order to apply this special initial conditions, we need to assume the following,

1. the colliding particles has so much energy that the flow of energy and matter after the collision remains unidirectional along the original collision axis; and
2. the transverse extent of the system is so large that the existence of the edge of matter in a direction transverse to the collision axis is of little relevance
3. Soon after the collision the baryon number of the nuclei begins to separate leaving the intermediate region a trail of energy, presumed to be in baryon-number-free QGP phase. The nuclei are trailed to right and left by the expansion of the energy they deposited at the instant of collision.

If all particles move with a constant velocity along a common longitudinal direction z in laboratory frame, the trajectory of each particle is a straight line $z = vt$, leading from the interaction point to the freeze-out location on the hyperbolic τ_f =constant surface.
Thus the initial conditions are;

- $$\epsilon = \epsilon_0(\tau, \eta_s) \tag{3.61}$$

- $$P = P_0(\tau, \eta_s) \tag{3.62}$$

- $$T = T_0(\tau, \eta_s) \tag{3.63}$$

- $$u^\mu = (u_0, \frac{z}{t}, 0, 0) \tag{3.64}$$

- $$\theta(\eta_s, \tau) = \eta_s \tag{3.65}$$

3.10.2 Solutions of $(0+1)$ hydrodynamics equation

$$0 = \tau\frac{\partial\epsilon}{\partial\tau} + \epsilon + P \tag{3.66}$$
$$0 = \frac{\partial P}{\partial\eta_s} \tag{3.67}$$
$$0 = \tau\frac{\partial n_{bar}}{\partial\tau} + n_{bar} \tag{3.68}$$
$$0 = c_s^2 + \tau\frac{\partial \ln T}{\partial\tau} \tag{3.69}$$

Thus, the solutions can be enumerated as follows,

$$n_{bar}(\tau) = n_{bar}(\tau_0)\frac{\tau_0}{\tau}, \tag{3.70}$$
$$s(\tau) = s(\tau_0)\frac{\tau_0}{\tau}, \tag{3.71}$$
$$\epsilon(\tau) = \epsilon_0\left(\frac{\tau_0}{\tau}\right)^{4/3}, \tag{3.72}$$
$$T(\tau) = T_0\left(\frac{\tau_0}{\tau}\right)^{1/3} \tag{3.73}$$

Here 1/3 in the exponent corresponds to the speed og sound in that medium, since we had taken ideal gas speed of sound is constant as the EoS is $\frac{dP}{d\epsilon} = c_s^2 = \frac{1}{3}$. The 4^{th} equation represents cooling of the QGP state, since decreases with τ. On numerical computation also the following results are obtained. I used 4^{th} Runge Kutta method to solve the differential equations and for the lattice EoS, I first used 4 point interpolation to correlate Pressure v/s energy density and then again used RK-4 to solve the corresponding differential equations.

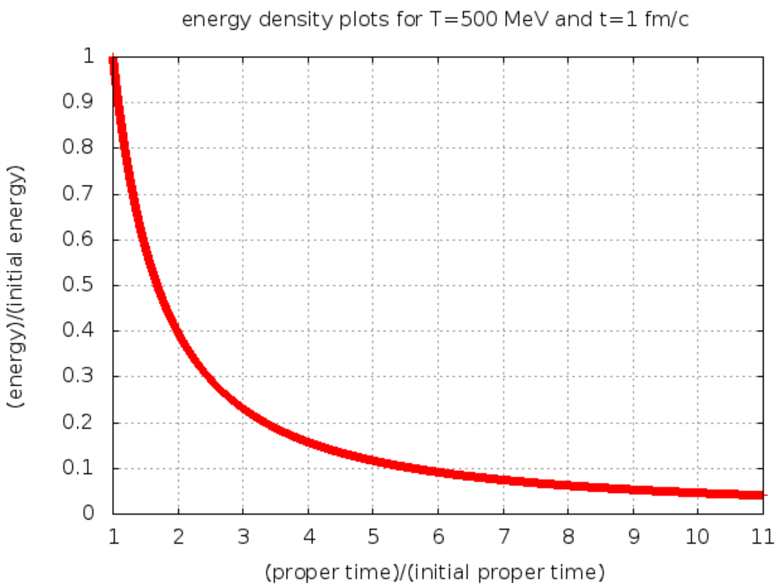

Figure 3.6: $\frac{\epsilon}{\epsilon_0} v/s \frac{\tau}{\tau_0}$ plot

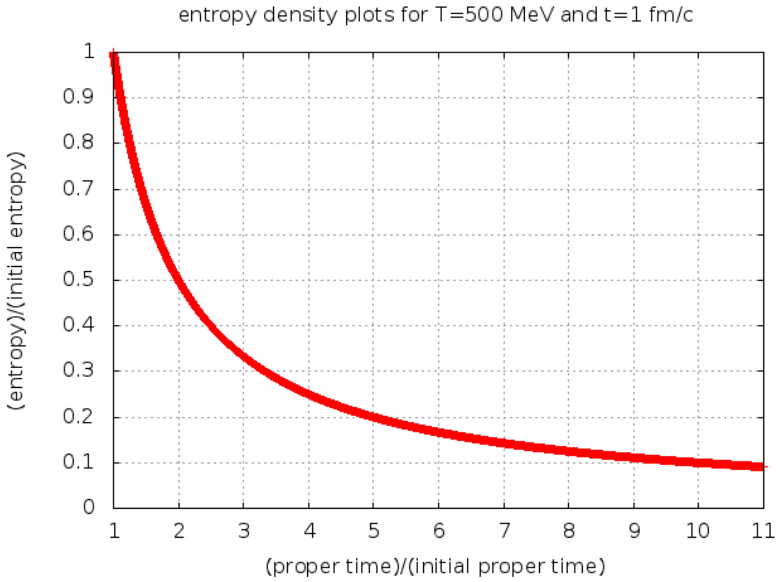

Figure 3.7: $\frac{s}{s_0} v/s \frac{\tau}{\tau_0}$ plot

3.10.3 Solution of Hydrodynamics Equation with given Lattice EoS

The given lattice EoS was

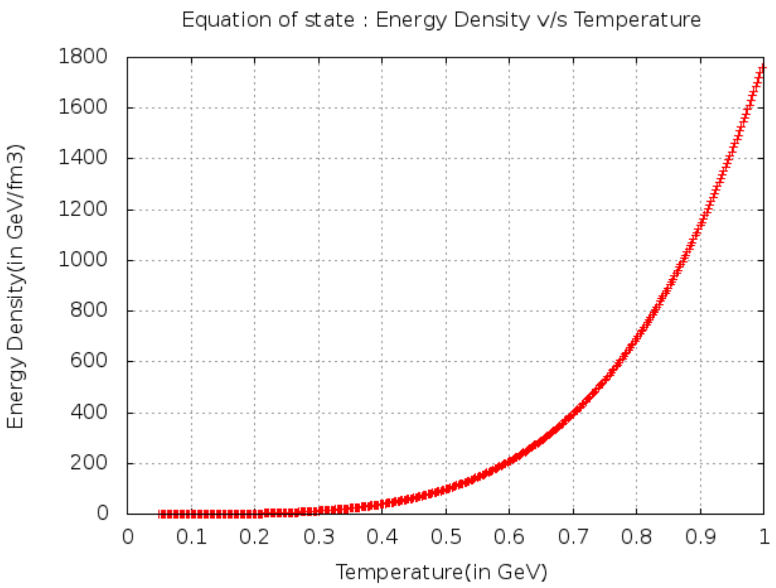

Figure 3.8: ϵ v/s T plot

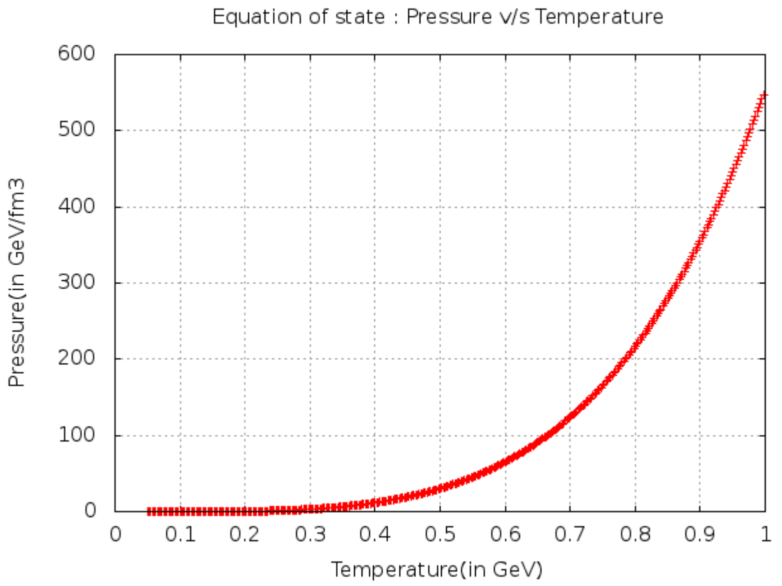

Figure 3.9: P v/s T plot

And the solution for energy density is,

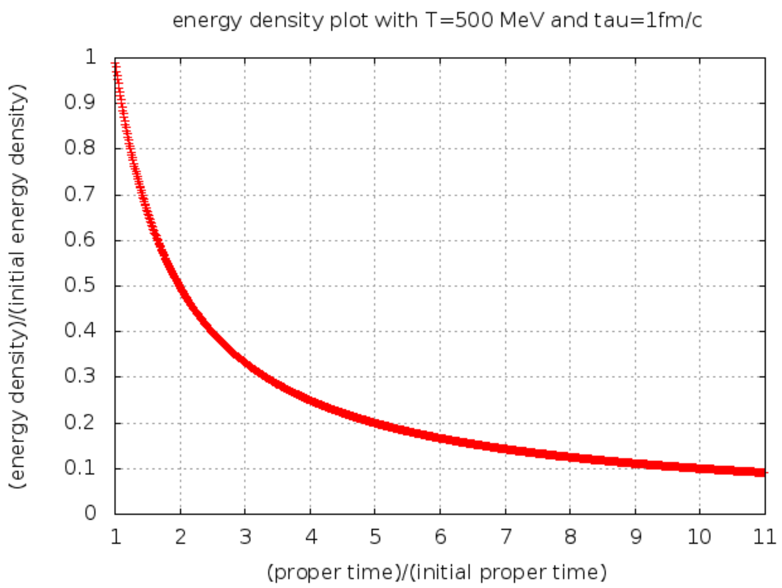

Figure 3.10: $\frac{\epsilon}{\epsilon_0} v/s \frac{\tau}{\tau_0}$ plot

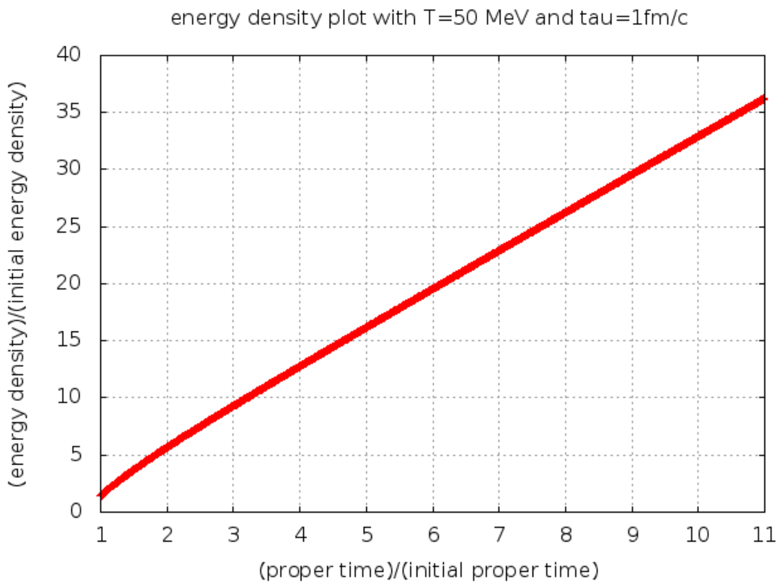

Figure 3.11: $\frac{\epsilon}{\epsilon_0} v/s \frac{\tau}{\tau_0}$ plot

APPENDIX A

User Interfaces