Q 1 Consider a system with four possible states 1, 2, 3, 4. Now consider three configurations A, B, C. For each configuration, that may correspond to a bulk energy level, we have a different set of probabilities for each state

Energy level A : $P_1 = 1$, $P_2 = 0$, $P_3 = 0$ & $P_4 = 0$

Energy level B : $P_1 = \frac{1}{2}$, $P_2 = \frac{1}{4}$, $P_3 = \frac{1}{8}$, $P_4 = 1/8$

Energy level C : $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{4}$, $P_3 = \frac{1}{4}$, $P_4 = 1/4$

Find the entropy of each level via

$$S = -K_B \sum_{n=1}^{4} P_n \ln P_n$$

Soln:

$$A: \quad S_A = -K_B \left(1 \ln 1 + 0 \ln 0 + 0 \ln 0 + 0 \ln 0\right) = 0$$

($\lim_{x \to 0} x \ln x = 0$)

$$B: \quad S_B = -K_B \left(\frac{1}{2} \ln \frac{1}{2} + \frac{1}{4} \ln \frac{1}{4} + \frac{1}{8} \ln \frac{1}{8} + \frac{1}{8} \ln \frac{1}{8}\right) = 1.21301\, K_B$$

$$C: \quad S_C = -K_B \left(\frac{1}{4} \ln \frac{1}{4} + \frac{1}{4} \ln \frac{1}{4} + \frac{1}{4} \ln \frac{1}{4} + \frac{1}{4} \ln \frac{1}{4}\right) = 1.38629\, K_B$$

- Least random is at energy level A. There is certainty that the configuration is state 1. There is no randomness to this and the entropy is formally zero. (Prob of finding a molecule in ground state is 1).
- Less random is B with some bias towards states 1 & 2. It has positive ~~energy~~ entropy.

c: has the highest entropy. In this configuration all states are equally likely. In this sense it is most random. ②

Q2. Same as ① for pair of ordinary dice.

P_n = probability of rolling each number

Roll	P_n	$-K_B P_n \ln P_n$
2	1/36	0.0995 KB
3	2/36	0.1605 KB
4	3/36	0.2071 KB
5	4/36	0.2441 KB
6	5/36	0.2742 KB
7	6/36	0.2986 KB
8	5/36	0.742 KB
9	4/36	0.2441 KB
10	3/36	0.2071 KB
11	2/36	0.1605 KB
12	1/36	0.0995 KB
Total	$\sum_n P_n = 1$	$\boxed{-K_B \sum_n P_n \ln P_n = 2.2694 K_B}$

If each roll were equally likely, then entropy is different. – need single dice to achieve this, which will be eleven sided.

Roll	P_n	$-K_B P_n \ln P_n$
2	1/11	$0.21799 K_B$
3	1/11	
4	1/11	"
5	1/11	
6	1/11	
7	1/11	
8	1/11	
9	1/11	
10	1/11	
11	1/11	
12	1/11	
Total	$\sum_n P_n = 1$	$-K_B \sum_n P_n \ln P_n = 2.3979 K_B$

2nd case more random than 1st case

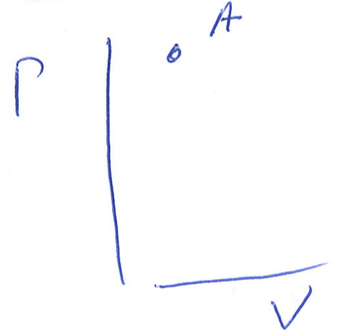
$S_{2nd} > S_{1st}$

Q3. One mole of an ideal monoatomic gas is the working substance of an ideal engine. The system is initially at point A with pressure P_A, volume V_A and temperature $T_A = T_H$. It is the allowed to expand isothermally to point B, the cool isochorically to point C, then contract isothermally to point D, then finally warm up isochorically back to the initial point A.

Let $P_A = 1 \times 10^5$ Pa ; $V_B =$ ~~306~~ $3 V_A$

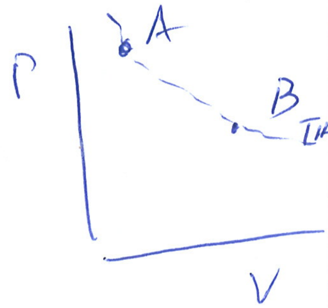
$T_H = 600K$ and $T_L = 300K$.

(a) What is the volume this gas occupies at A?



$$PV = nRT \Rightarrow P_A V_A = nRT_A$$

$$\Rightarrow V_A = \frac{nRT_A}{P_A}$$



(b) What is the work done by the gas W_{AB}

* How can it isothermally go from A to B

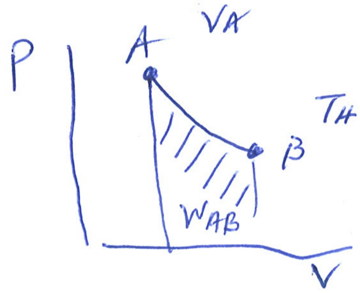
$PV = nRT \Rightarrow$ P and V must change to keep nRT_H constant

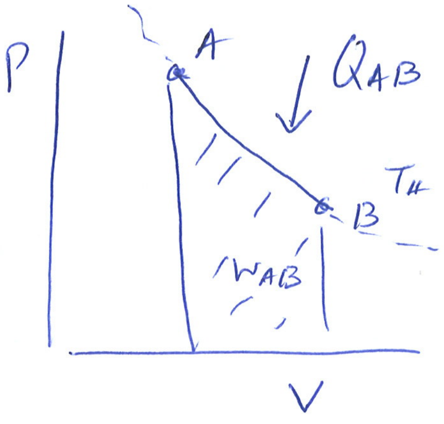
* If V increases, the gas does positive work (energy leaves the gas)

$$PV = nRT \Rightarrow P = \frac{nRT}{V}$$

$$W_{AB} = \int_{V_A}^{V_B} P dV = \int_{V_A}^{V_B} \frac{nRT_H}{V} dV = nRT_H \int_{V_A}^{V_B} \frac{dV}{V}$$

$$= nRT_H \ln(V_B/V_A)$$





* Since internal energy does not change, energy (heat) must be added to compensate for the work gas does

$$\Delta U = Q_{AB} - W_{AB} = 0$$

$$\Rightarrow Q_{AB} = W_{AB}$$

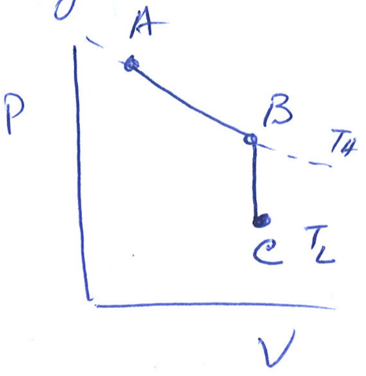
* What is the heat added to the gas Q_{AB} in going from A to B - $Q_{AB} = nRT_H \ln(V_B/V_A)$

* What is the change in entropy of the gas ΔS_{AB}

$$\Delta S_{AB} = \int_A^B \frac{dQ}{T} = \frac{1}{T_H}\int_A^B dQ = \frac{Q_{AB}}{T_H}$$

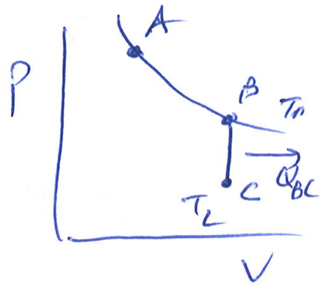
$$\boxed{\Delta S_{AB} = nR\ln\left(\frac{V_B}{V_A}\right)} > 0$$

(c) What is the work done by the gas W_{BC} in going from B to C



- No volume change $\Rightarrow W_{BC} = 0$
- This does not mean energy didnot change form
- How did the gas temperature drop with no change in volume.

— Energy has to leave the system



$$\Delta U = \frac{3}{2} nR\Delta T = Q_{BC} - \cancelto{0}{W_{BC}}$$

$\frac{3}{2}$ = mono atomic gas

$$\Rightarrow Q_{BC} = \frac{3}{2}nR(T_L - T_H) < 0$$

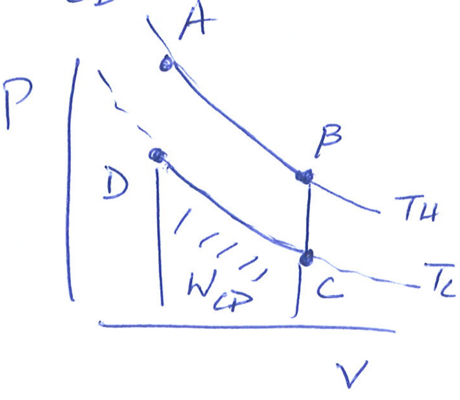
What is the change in entropy going from B to C

$$\Delta S_{BC} = \int_B^C \frac{dQ}{dT} = \int_{T_H}^{T_L} \frac{\frac{3}{2}nR\,dT}{T} = \frac{3}{2}nR\ln(T_L/T_H)$$

$$\Delta S_{BC} = \frac{3}{2} nR \ln\left(\frac{T_L}{T_H}\right)$$

$$= -\frac{3}{2} nR \ln\left(\frac{T_H}{T_L}\right) < 0$$

(d) What is the work done by the gas W_{CD}

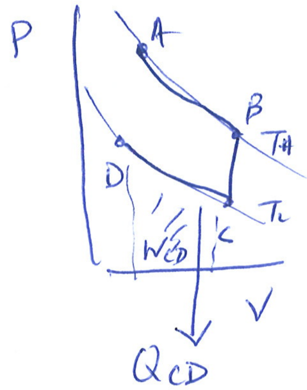


* V decreases, gas does negative work. $V_C = V_B$ and $V_D = V_A$

$$W_{CD} = \int_{V_C}^{V_D} P\,dV = \int_{V_C}^{V_D} \frac{nRT_L}{V}\,dV$$

$$= nRT_L \int_{V_B}^{V_A} \frac{dV}{V} = nRT_L \ln(V_A/V_B)$$

$$= -nRT_L \ln(V_B/V_A) < 0$$



* How can it isothermally go from C to D
Internal energy doesn't change, energy (heat) must leave the gas as volume decreases

$$\Delta U = Q_{CD} - W_{CD} = 0$$

$$\Rightarrow Q_{CD} = W_{CD} = -nRT_L \ln(V_B/V_A)$$

* What is the heat leaving the gas Q_{CD}

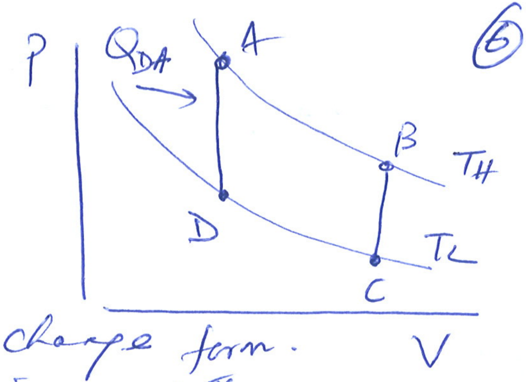
$$Q_{CD} = -nRT_L \ln(V_B/V_A)$$

* What is the change in entropy ΔS_{CD}

$$\Delta S_{CD} = \int_C^D \frac{dQ}{T} = \frac{1}{T_L}\int_C^D dQ = \frac{Q_{CD}}{T_L}$$

$$\boxed{\Delta S_{CD} = -nR \ln(V_B/V_A) < 0}$$

(e) What is the work done by gas W_{DA}



- volume change not there

  Work = 0

- Does that mean energy did not change form.

How could gas temperature increase with the no change in volume.

- Energy has to enter the system

- Mono atomic gas $\Delta U = \frac{3}{2} nR\Delta T = Q_{DA} - \cancel{W_{DA}}^{\to 0}$

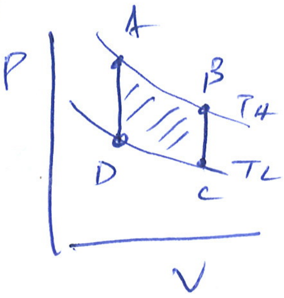
$$Q_{DA} = \frac{3}{2} nR(T_H - T_L) > 0$$

+ what is the change in entropy ΔS_{DA}

$$\Delta S_{DA} = \int_D^A \frac{dQ}{T} = \int_{T_L}^{T_H} \frac{\frac{3}{2} nR\,dT}{T} = \frac{3}{2} nR \ln(T_H/T_L)$$

$$\boxed{\Delta S_{DA} > 0}$$

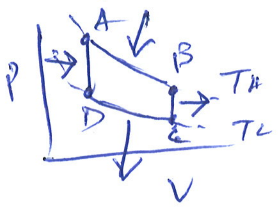
(f) what is the net work done



$$W_{net} = \underbrace{W_{AB}}_{nRT_H \ln\left(\frac{V_B}{V_A}\right)} + \cancel{W_{BC}}^{\to 0} + \underbrace{W_{CD}}_{-nRT_L \ln\left(\frac{V_B}{V_A}\right)} + \cancel{W_{DA}}^{\to 0}$$

$$W_{net} = nR(T_H - T_L)\ln\left(\frac{V_B}{V_A}\right)$$

(g) what is the net heat transfered?



$Q_{net} = Q_{AB} + Q_{BC} + Q_{CD} + Q_{DA}$ or

$\Delta U = 0$ (cyclic) | $\boxed{Q_{net} = W_{net} = nR(T_H - T_L)\ln\frac{V_B}{V_A}}$

$$Q_{net} = \underbrace{Q_{AB}}_{nRT_H \ln\left(\frac{V_B}{V_A}\right)} + \underbrace{\cancel{Q_{BC}}}_{\frac{3}{2} nR(T_L - T_H)} + \underbrace{Q_{CD}}_{-nRT_L \ln\left(\frac{V_B}{V_A}\right)} + \underbrace{\cancel{Q_{DA}}}_{\frac{3}{2} nR(T_H - T_L)}$$

$$\boxed{Q_{net} = nR(T_H - T_L)\ln\left(\frac{V_B}{V_A}\right)}$$

(h) net entropy change in one cycle?

easy $\Delta S\Big|_{cycle} = 0$!

OR

$$\Delta S_{net} = \Delta S_{AB} + \Delta S_{BC} + \Delta S_{CD} + \Delta S_{DA}$$

$$\cancel{nR \ln\left(\frac{V_B}{V_A}\right)} \qquad \cancel{-\tfrac{3}{2} nR \ln\left(T_H/T_L\right)} \qquad \cancel{-nR \ln\left(\frac{V_B}{V_A}\right)} \qquad \cancel{\tfrac{3}{2} nR \ln\left(T_H/T_L\right)}$$

$= 0$