Introduction to the spherical co-ordinate system

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Spherical co-ordinate system

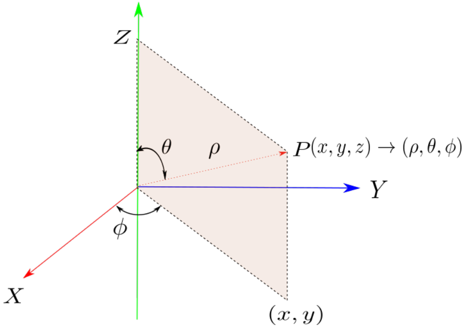
Idea: Convert the representation of a point (x, y, z) to (ρ, θ, ϕ).

Spherical coordinates

$\rho =$ Distance from 0 to P

$\phi =$ Angle between (x, y) and X-axis

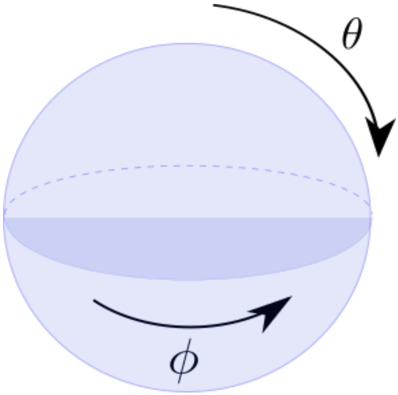
$\theta =$ Angle between (x, y, z) and Z-axis



Visualization

Visualize:
ϕ = longitude
θ = latitude

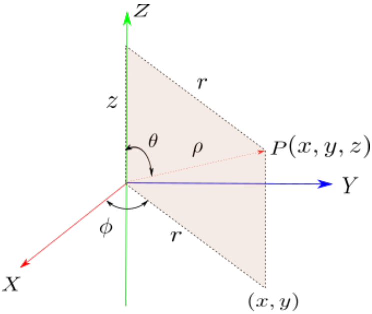
Constraints:
$\rho \geq 0$
$0 \leq \theta \leq \pi$
$0 \leq \phi \leq 2\pi$
$x^2 + y^2 + z^2 = \rho^2$



Example 1

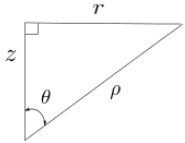
Let us find equations for x, y, z in terms of ρ, θ, ϕ

$$\begin{aligned} z &= \rho \cos\theta \\ r &= \rho \sin\theta \end{aligned}$$



So,

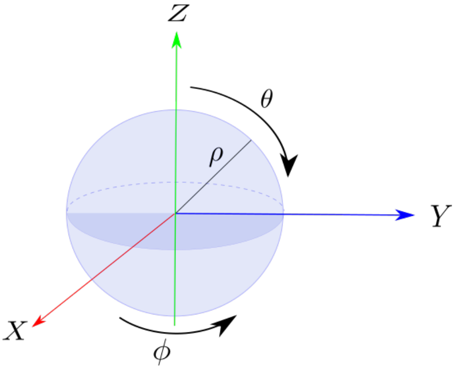
$$\begin{aligned} x &= r\ \cos\phi &&= \rho \sin\theta \cos\phi \\ y &= r\ \sin\phi &&= \rho \sin\theta \sin\phi \\ z &= \rho\cos\theta \end{aligned}$$



Example 2

Find the volume of a ball of radius R

$$V = \iiint dx\, dy\, dz$$



Spherical co-ordinates great for sphere and cones

Constraints:

$$0 \leq \rho \leq R$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq 2\pi$$

Jacobian: $(x, y, z) \to (u, v, w)$

$$dV = dxdydz = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\,dv\,dw = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} du\,dv\,dw$$

$$x = \rho \sin\theta \cos\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\theta$$

$$\frac{\partial x}{\partial \rho} = \cos\phi \sin\theta \qquad \frac{\partial x}{\partial \theta} = \rho \cos\theta \cos\phi \qquad \frac{\partial x}{\partial \phi} = -\rho \sin\theta \sin\phi$$
$$\frac{\partial y}{\partial \rho} = \sin\theta \sin\phi \qquad \frac{\partial y}{\partial \theta} = \rho \cos\theta \sin\phi \qquad \frac{\partial y}{\partial \phi} = \rho \sin\theta \cos\phi$$
$$\frac{\partial z}{\partial \rho} = \cos\theta \qquad \frac{\partial z}{\partial \theta} = -\rho \sin\theta \qquad \frac{\partial z}{\partial \phi} = 0$$

Jacobian :

$$\begin{vmatrix} \cos\phi\sin\theta & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix}$$
$$= \rho^2\sin\theta$$

So we have,

$$dV = dxdydz = \rho^2\sin\theta \; d\rho\, d\theta\, d\phi$$

$$V = \iiint dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\theta d\rho d\theta d\phi$$

$$\boxed{= \int_0^R \rho^2 d\rho \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{4}{3}\pi R^3}$$