Generalization of Thermodynamic Square

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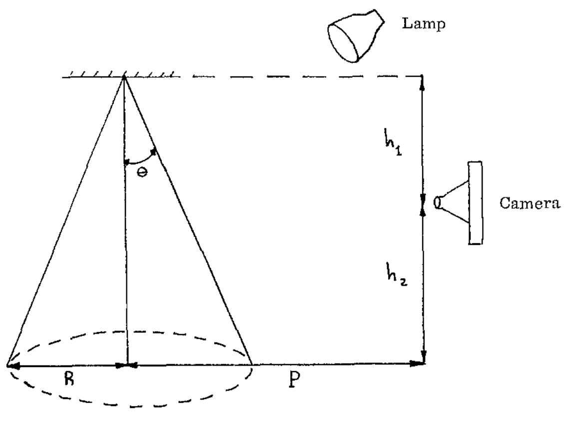

FIG. 1. Experimental arrangement (not to scale; see Ref. 6).

circle. We check this by watching for any precession as the radius of the circle reduces with time. The students photograph the motion of the pendulum from the side with the strobe camera for slightly more than one cycle, with the region of overlap of the series of exposures occurring when the bob is nearest the camera. The camera has been positioned carefully by the instructor to minimize optical distortion in the half angle of the cone θ, viz., with its film plane vertical and lens slightly above the center of the thread.[6] Figure 2 is an actual photo obtained as described above with speed 3000 film, lens setting f/5.6, shutter setting on bulb, and strobe shutter set at 1/60 sec.

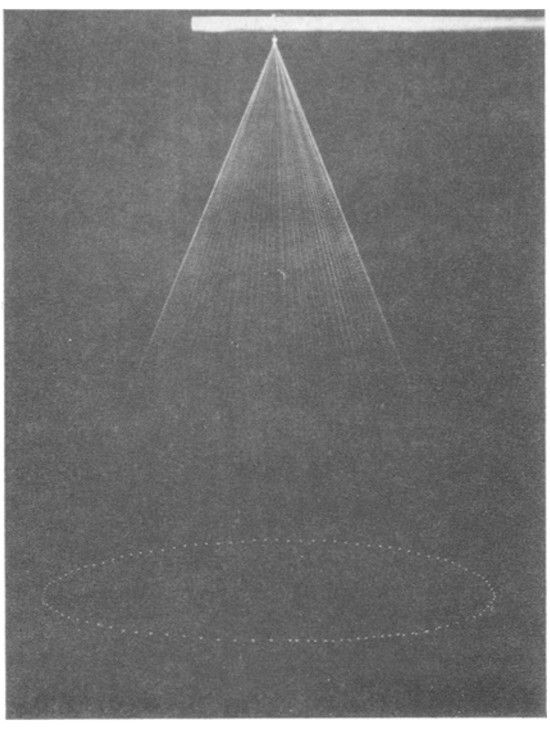

FIG. 2. Actual photo of conical pendulum.

In the laboratory frame, two forces act on the ball: the tension in the string and the weight of the ball. Because the motion of the ball occurs in a horizontal plane, the vertical components of the two forces must be equal and opposite, i.e., $F\ \cos\theta = mg$; but there is an unbalanced horizontal force $F_c = F\ \sin\theta = mg\ \tan\theta$. The resulting motion of the ball in the plane is observed to be uniformly circular, so that F_c should be equal to mv^2/R.

From the photograph the student measures the angle of the cone 2θ, either by projecting the film[2] if Polaroid projection film has been used, or by using a measuring eyepiece[7] if Polaroid speed 3000 film has been used, and counts the number of exposures in one cycle to find the period T. He then calculates and compares $mg\ \tan\theta$ with $mR\omega^2 = 4\pi^2 ml\ \sin\theta T^{-2}$.

Alternatively, one may determine g from this experiment.[8] Solving for g leads to

$$g = \omega^2 l\ \cos\theta \qquad (l\ \cos\theta \text{ is the height}).$$

The principal source of error in this experiment seems to be in connection with the angle θ, partly because the orbit may not be a true circle and partly because it is difficult to measure 2θ to better than $\frac{1}{2}$ deg. T can be determined to better than $\frac{1}{2}\%$ by counting the exposures in one revolution and interpolating between the overlapped exposures. With careful work an over-all error of $\approx 1\%$ is possible.

1 We use the MCI Kinematics-Dynamics Camera attachment to provide the stroboscopic effect. The attachment fits our Polaroid Model 110B camera. It contains a 600-rpm synchronous motor driving a shutter disk. Three shutter disks were provided: one precut for 60 exposures/sec and two blanks. (Available from MCI Inc., 2324 First St., Livermore, Calif. 94550.)

2 D. L. Enlow and P. A. Schroeder, Am. J. Phys. **35**, 651 (1967).

3 E. Huggins, *Physics I* (W. A. Benjamin Company, Inc., N. Y., 1965).

4 W. E. Hazen *et al.*, Am. J. Phys. **37**, 174 (1959).

5 W. W. McCormick, *Laboratory Experiments in Physics* (MacMillan Co., N. Y., 1966).

6 Prof. B. Stewart of the MSU Mathematics Department first showed us that for the film plane vertical, the angle on the film will be the same as the true angle of the cone provided $h1/h2 = P/(P^2 - R^2)^{1/2}$.

7 Bausch and Lomb 7× measuring magnifier with protractor scale.

8 F. Wunderlich, Am. J. Phys. **34**, 1199 (1966).

Generalization of Thermodynamic Square

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The thermodynamic square[1] provides a convenient reminder of the Maxwell relations, which are available for transforming partial derivatives in thermodynamics. Any given square, however, deals only with two thermodynamic degrees of freedom.

When there are three degrees of freedom, one may draw three squares, one for each pair of degrees of freedom.

There is, however, a single figure, namely the octahedron, which can incorporate all 12 Maxwell relations in a natural and convenient manner. For example, suppose that the differential of the energy is given by the expression

$$dU = TdS - pdV + Hdm$$

and that the differentials of the seven other potentials are

$$dU' = TdS - pdV - mdH,$$

$$dE = TdS + Vdp + Hdm,$$

$$dE' = TdS + Vdp - mdH,$$

$$dF = -SdT - pdV + Hdm,$$

$$dF' = -SdT - pdV - mdH,$$

$$dG = -SdT + Vdp + Hdm,$$

$$dG' = -SdT + Vdp - mdH.$$

Then the associated thermodynamic octahedron is as shown in Fig. 1.

To construct the octahedron, one first assigns the variables to vertices so that conjugate variables occupy opposite vertices. One then assigns each potential to that face which is bounded by the three variables on which it depends. Each variable in the figure is assigned the sign which it bears in the above equations (i.e., in those in which it does not occur as a differential).

One may note that the octahedron contains three squares and that these are the three thermodynamic squares alluded to above. The Maxwell relations may be read off in the usual manner; for instance

$$\partial m/\partial S\,|_H = -\partial T/\partial H\,|_S,$$

where on both sides of the equation we also hold either V or p constant (depending on whether we assume the Maxwell relation to have arisen from differentiation of U' or of E').

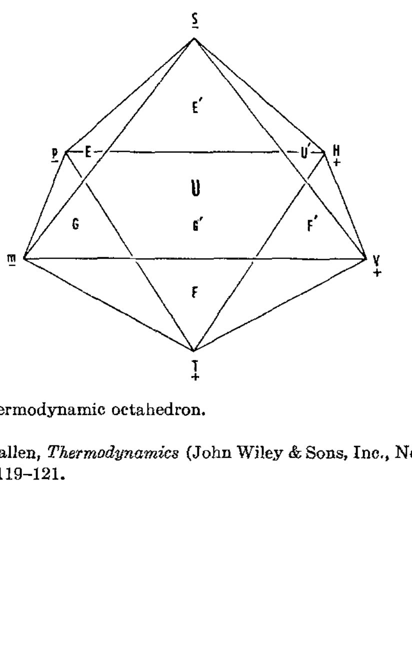

FIG. 1. Thermodynamic octahedron.

[1] H. B. Callen, *Thermodynamics* (John Wiley & Sons, Inc., New York, 1960), pp. 119–121.

Comments on Generalized Mechanics

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Koestler and Smith,[1] extending the work of Borneas,[2] attempted recently at a generalization of classical Hamiltonian mechanics, based on a Lagrangian

$$L(t,\, q_k,\, \cdots,\, q_k^{(m)},\, \cdots\, q_k^{(s)}) \qquad k=1,2,\cdots,r \quad m=0,1,\cdots,s, \tag{1}$$

which is dependent on the time, the generalized coordinates $q_k(t)$ and their higher derivatives $q_k^{(m)}(t)$.

The first-cited authors stress repeatedly "the lack of a fundamental definition relating canonically conjugate coordinates and momenta for this generalized mechanics." In their work "it appears impossible to define such coordinates and momenta in a general sense."

These statements contradict the fact that a consistent theory of a generalized mechanics was fully developed at early as 1848 by Ostrogradsky.[3] A very convenient way to realize which quantities should be considered as canonically conjugate follows from the variation of the action function, including the variation of the end points. Indeed, after successive integrations, the latter can be put in the form

$$\delta\int_A^B L dt \equiv \int_A^B p_k^{-1}\delta q_k dt + \sum_{k=1}^{r}\sum_{m=0}^{s-1}[p_k^m \delta q_k^{(m)}]_A^B, \tag{2}$$

which indicates that the rs independent variables

$$p_k^m \equiv \sum_{j=0}^{s-m-1}(-1)^j\left(\frac{\partial}{\partial t}\right)^j \frac{\partial L}{\partial q_k^{j+m+1}} \qquad \begin{matrix} k=1,2,\cdots,r \\ m=0,1,\cdots,s-1 \end{matrix} \tag{3}$$

are the momenta conjugate to the rs coordinates $q_k^m = q_k^{(m)}$. Extending the notation for $m=-1$, the r Euler equations can be written:

$$p_k^{-1} \equiv \sum_{j=0}^{s}(-\partial/\partial t)^j(\partial L/\partial q_k^j) = 0 \qquad k=1,2,\cdots,r. \tag{4}$$

The reduction of the Euler equations to a linear Hamiltonian system of order $2rs$ can be found, e.g., in the standard treatise of Whittaker,[4] where the labeling $q_k^m = q_{k+m}$, $p_k^m = p_{k+m}$ places all the generalized coordinates as well as the generalized momenta on the same footing. From this it is obvious that all the techniques of the Hamiltonian theory can be applied in a phase space of $2rs$ dimensions. In particular, the Poisson brackets are

$$[u, v] \equiv \sum_{k=1}^{r}\sum_{m=0}^{s-1}\left(\frac{\partial u}{\partial q_k^m}\frac{\partial v}{\partial p_k^m} - \frac{\partial u}{\partial p_k^m}\frac{\partial v}{\partial q_k^m}\right). \tag{5}$$

The treatment of Koestler and Smith is induced by an inadequate labeling of the canonical variables, resulting in