Greedy Algorithms, Matroids, and Parallel Complextiy

Joint work with Sumanta Ghosh and Roshan Raj

NISER Bhubaneswar

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Outline

- Introduction to matroids and connection with greedy algorithms
- Search vs. decision: parallel complexity
- Matroid intersection: deterministic parallel search to decision reduction

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- Search vs. decision: parallel complexity
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Takeaways:

- Isolation Lemma
- Succinct representation of all MSTs
- Succinct representation of all maximum weight perfect matchings

Maximum weight spanning tree

Kruskal's Algorithm

- Sort the edges in decreasing order of weights.
- Keep selecting edges which do not create a cycle (maintain a forest).

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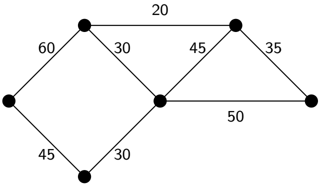

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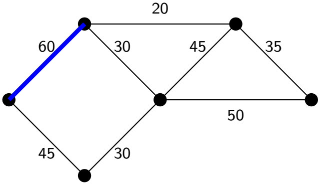

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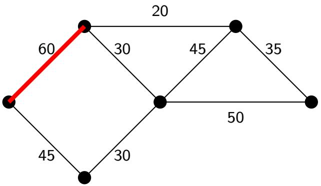

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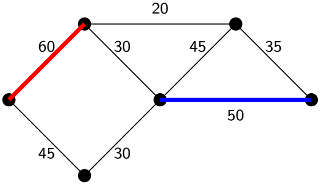

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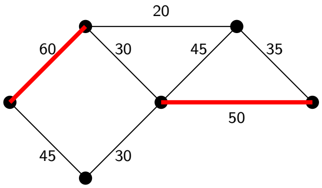

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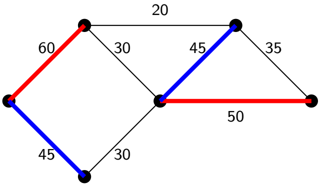

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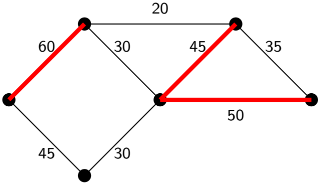

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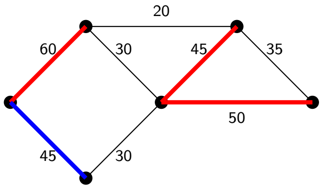

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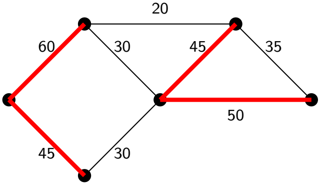

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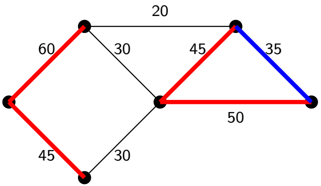

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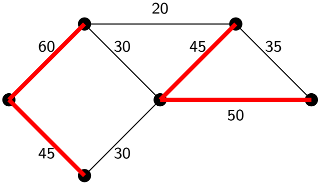

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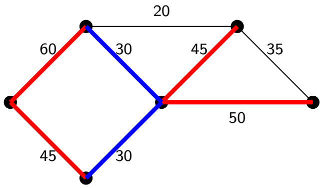

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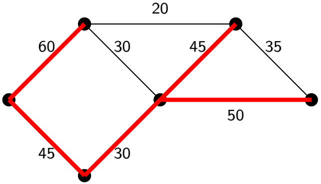

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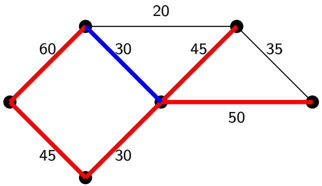

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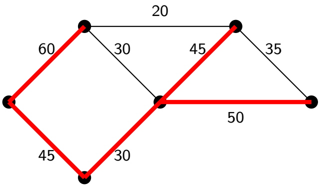

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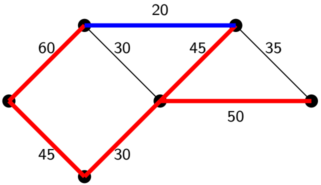

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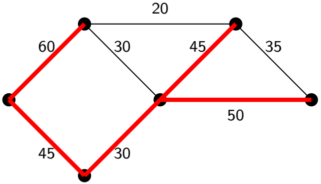

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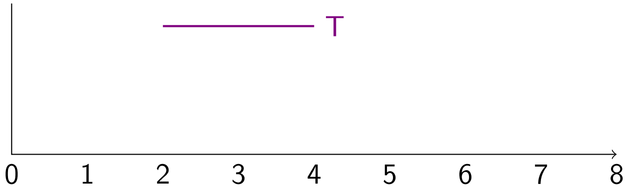

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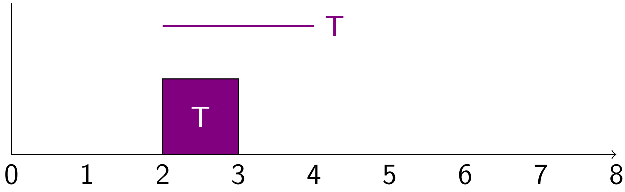

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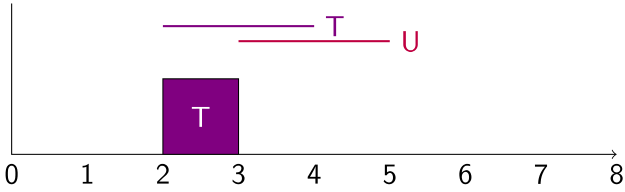

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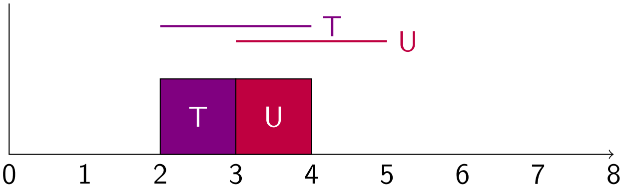

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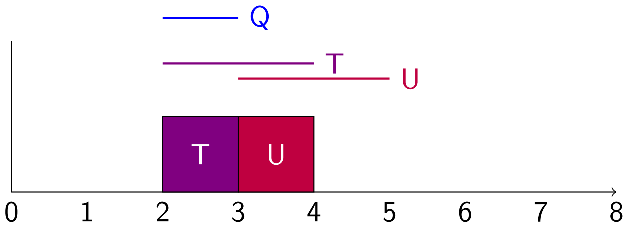

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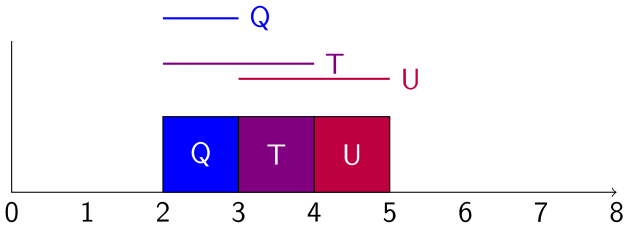

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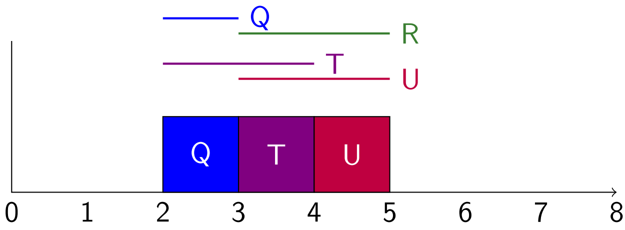

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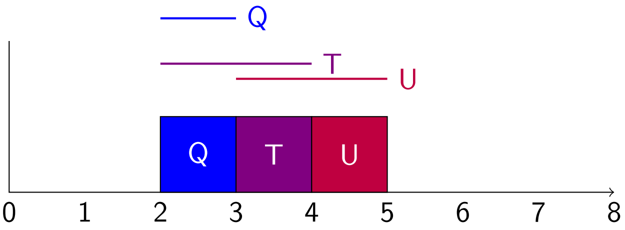

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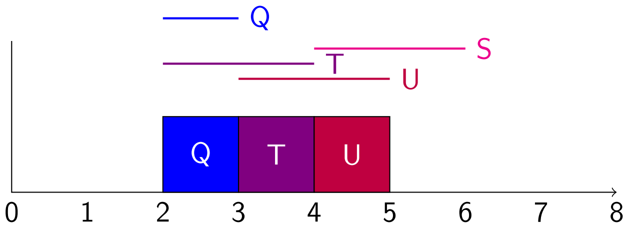

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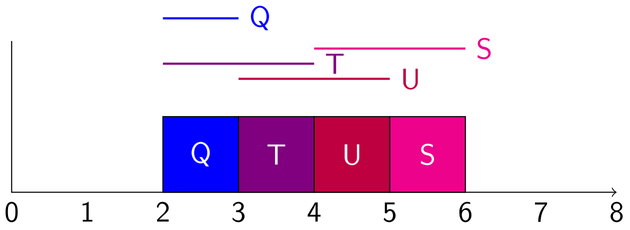

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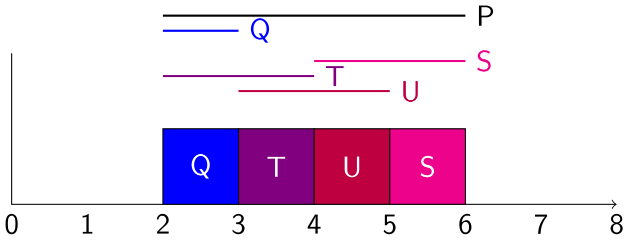

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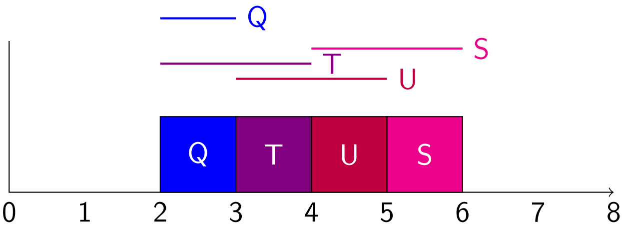

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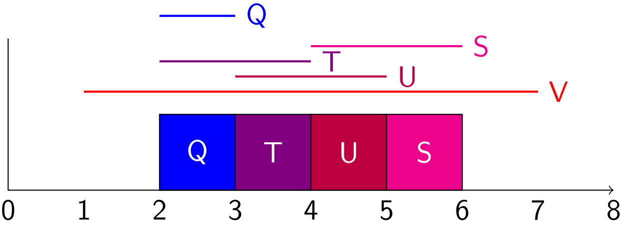

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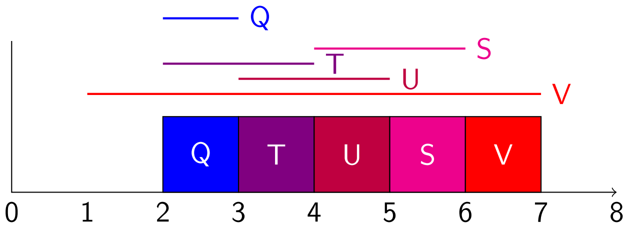

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  - $A, B \in \mathcal{I}$ with $|A| < |B|$ then $\exists a \in B \setminus A$ such that $A + a \in \mathcal{I}$.

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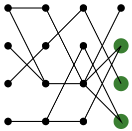

Examples of Matroids

- Graphic matroids $E \leftarrow$ edge set, $\mathcal{I} \leftarrow$ family of all forests.
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- Linear matroids
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  $E \leftarrow$ set of students in a college,
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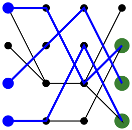

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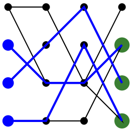

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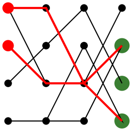

Matroids in Computer Science

- Combinatorial optimization
- Game theory
- Online algorithms
- Algebraic problems

Matroid Intersection

Problem

- Given two matroids on the same ground set,
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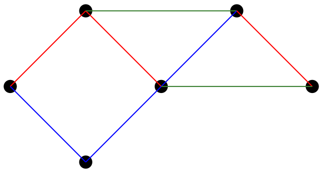

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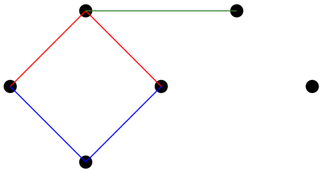

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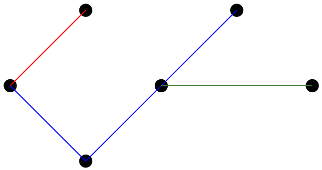

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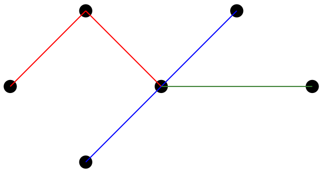

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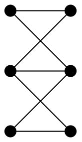

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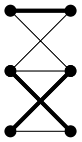

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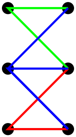

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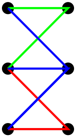

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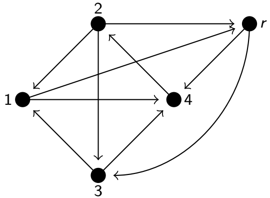

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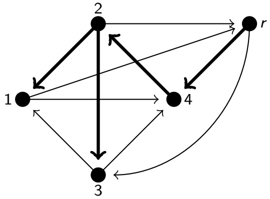

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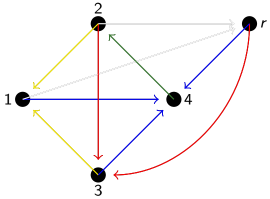

Figure: Graphic Matroid ∩ Partition Matroid

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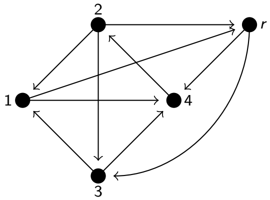

Figure: Graphic Matroid $\cap$ Partition Matroid

- At most 1 incoming edge at each vertex.

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- Rainbow spanning tree
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- Finding two disjoint spanning trees (Homework)

Search vs. Decision

Search vs. Decision

Satisfiability

- Decision: Given a Boolean formula, is there a satisfying assignment?
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- Is Search as easy as Decision?

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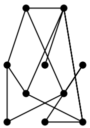

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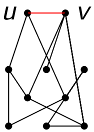

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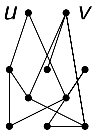

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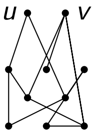

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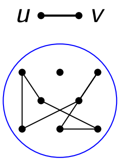

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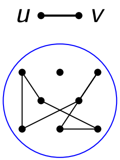

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- [Karp, Upfal, Wigderson 1985] studied this question, motivated by the parallel complexity status of matching and matroid intersection.

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Parallel Complexity of Matroid Intersection

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  Some exciting progress recently.

Search to weighted-decision

Search to weighted-decision deterministic parallel reduction

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Search to weighted-decision

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- If yes, discard e.

Search to weighted-decision: Randomized Reduction

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Search to weighted-decision: Randomized Reduction

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Search to weighted-decision: Randomized Reduction

- [MVV87] Isolating weight assignment: a weight assignment on the edges such that there is only one maximum weight perfect matching.
- Can we find unique max weight PM using non-adaptive weighted-decision queries?

Finding unique max weight perfect matching

- Find w^* using weighted-decision queries.
- For each edge e, in parallel:
  does $G - e$ have a perfect matching with weight $\geq w^*$?
  - If no, select e.
  - If yes, discard e.

Search to weighted-decision: Randomized Reduction

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Wake up!

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- Works for an arbitrary family of sets.
- Derandomizing Isolation Lemma remains an open question.

Search to weighted-decision: Deterministic Reduction

Main technical result

- Given two matroids,
- construct a weight assignment such that there is only one max weight common base (rainbow spanning tree)
- using $O(\log^2 n)$ rounds of weighted-decision queries.

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Algorithm at a high level

$S \leftarrow$ set of all common bases
while $(|S| > 1)$
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- In $O(\log n)$ rounds, unique max weight common base.
- Crucially use a succinct representation of the set of max weight common bases.

Succinct representation of all MSTs

- First question: How do we succinctly represent all maximum weight bases of a Matroid?
- How do we succinctly represent all maximum weight spanning trees in a graph?

Succinct representation of all MSTs

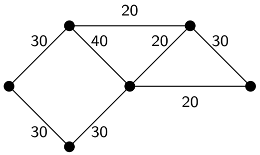
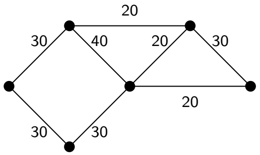


Figure: Graph G

Succinct representation of all MSTs



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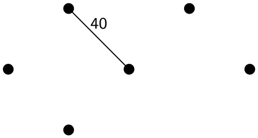

Figure: Graph G_{40}

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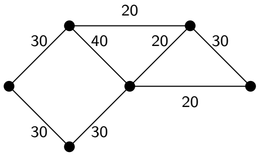

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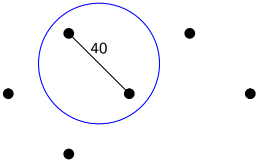

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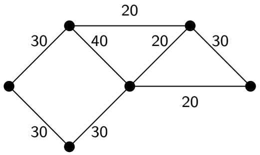

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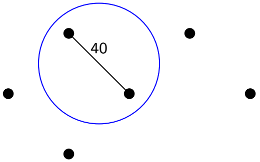

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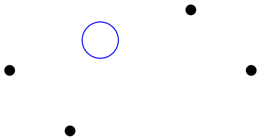

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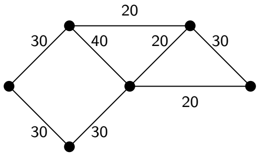

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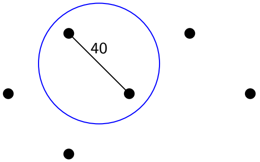

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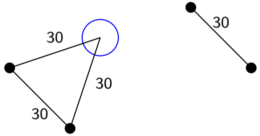

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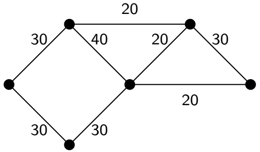

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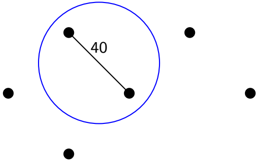

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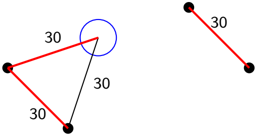

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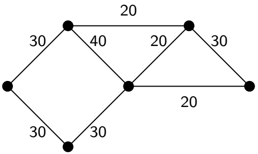

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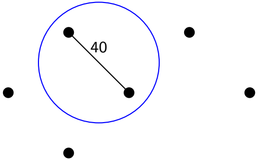

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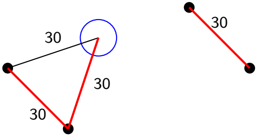

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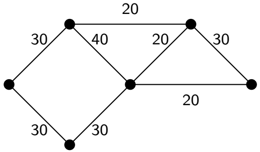

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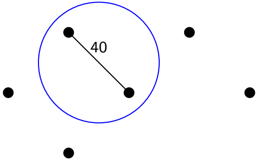

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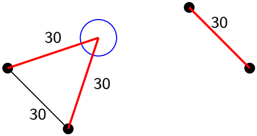

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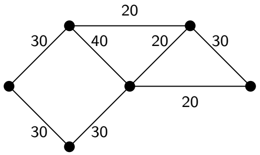

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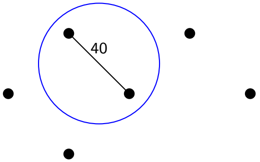

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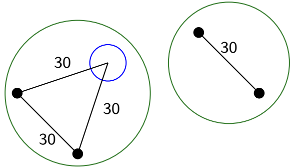

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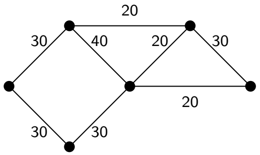

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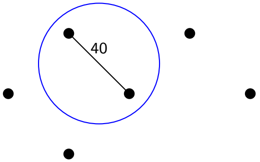

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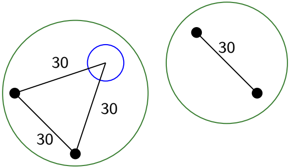
Figure: Graph G_{20}



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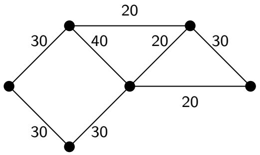

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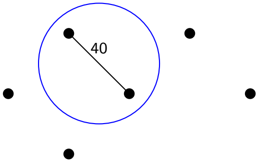

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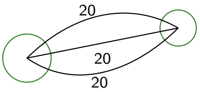

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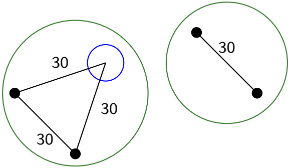

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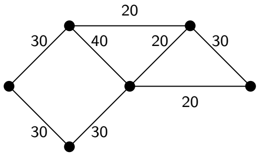

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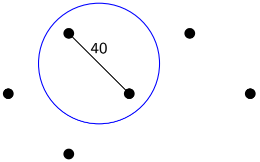

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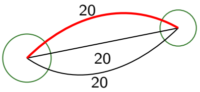

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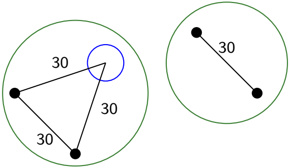

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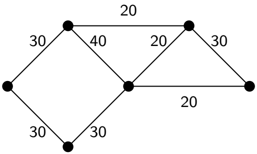

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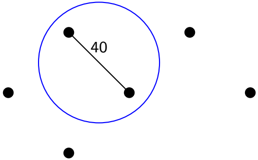

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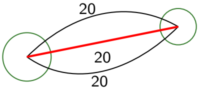

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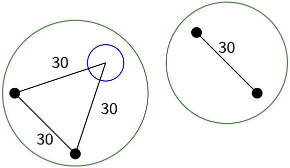

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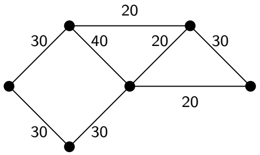

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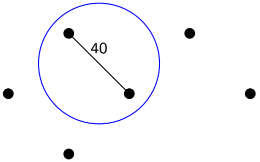

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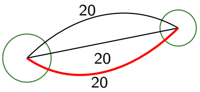

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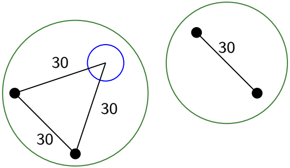

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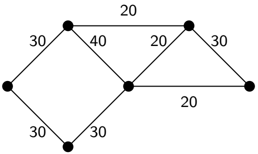
Succinct representation of all MSTs



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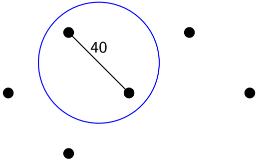

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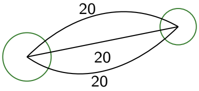

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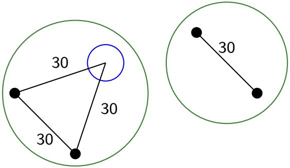

Figure: Graph G_{30}

MSTs in G = {largest forests in G_{40}} $\times$ {largest forests in G_{30}} $\times$ {largest forests in G_{20}}

Succinct representation of all MSTs

Observation: Every MST takes

- 1 edge from G_{40}
- 3 edges from G_{30}
- and 1 edge from G_{20}.

Succinct representation of all max weight common bases

Weight splitting theorem

Succinct representation of all max weight common bases

Weight splitting theorem

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Succinct representation of all max weight common bases

Weight splitting theorem

- Given two matroids M_1 and M_2 with a weight assignment w,
- there exists a weight splitting $w = w_1 + w_2$ such that
- set of max weight common bases =

$$\{\text{max weight bases in } M_1 \text{ w.r.t. } w_1\}$$
$$\cap$$
$$\{\text{max weight bases in } M_2 \text{ w.r.t. } w_2\}$$

Weight-splitting for Bipartite Perfect Matching

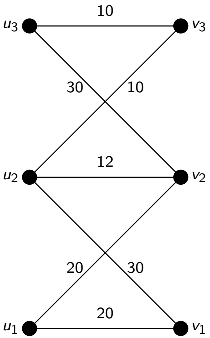


Figure: Weight-splitting

Three perfect matchings

Weight-splitting for Bipartite Perfect Matching

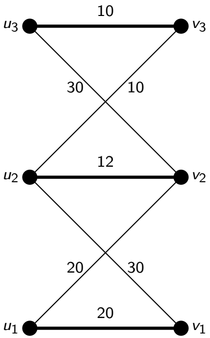


Figure: Weight-splitting

Three perfect matchings

- 10+12+20=42

Weight-splitting for Bipartite Perfect Matching

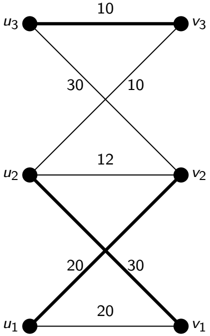
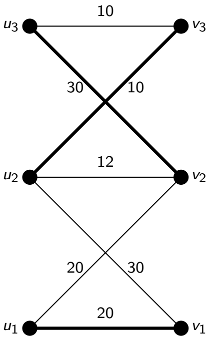


Figure: Weight-splitting

Three perfect matchings

- 10+12+20=42
- 10+20+30=60

Weight-splitting for Bipartite Perfect Matching



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Three perfect matchings

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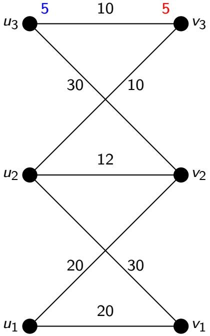

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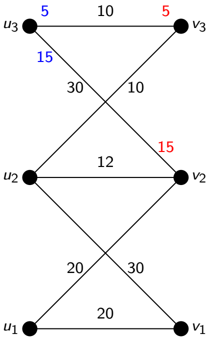

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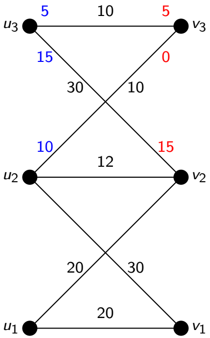

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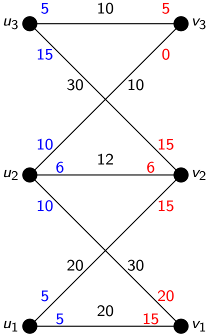

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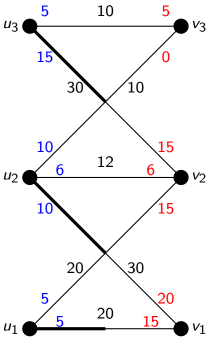

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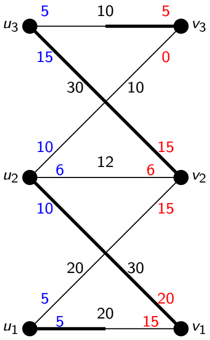
Three perfect matchings

- $10+12+20=42$
- $10+20+30=60$
- $30+10+20=60$

Weight-splitting for Bipartite Perfect Matching



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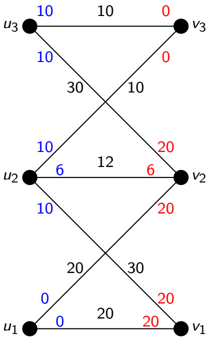

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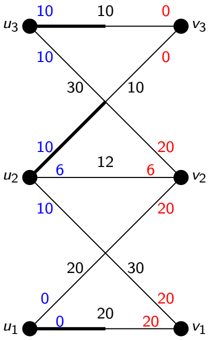

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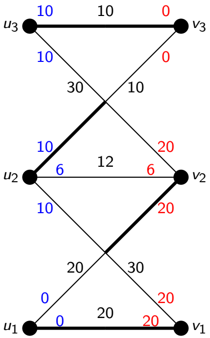

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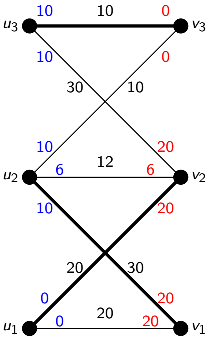

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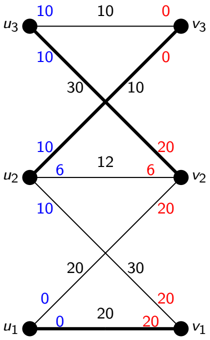

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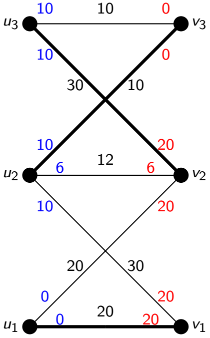

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Obs: A perfect matching maximizes w_1 and w_2 $\implies$ it maximizes $w_1 + w_2$.

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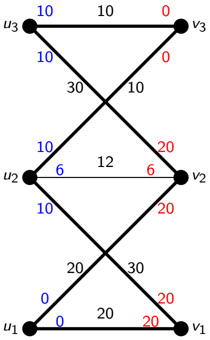

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Thm: All maximum weight perfect matchings can be obtained this way.

Ideas

Algorithm at a high level

$S \leftarrow$ set of all common bases
while $(|S| > 1)$
    Update w to enforce some tie breaks in S.
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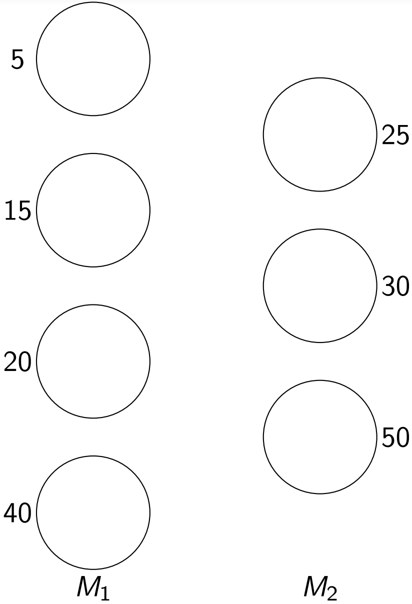
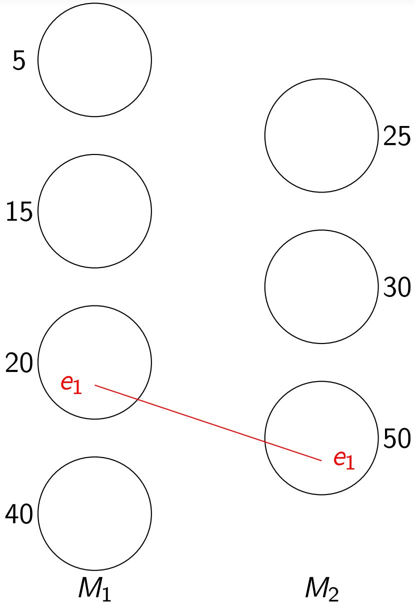
- How do we update w to break ties?
- Consider two max weight common bases and their symmetric difference.



Consider two max weight common bases.

Figure: A cycle in two common bases



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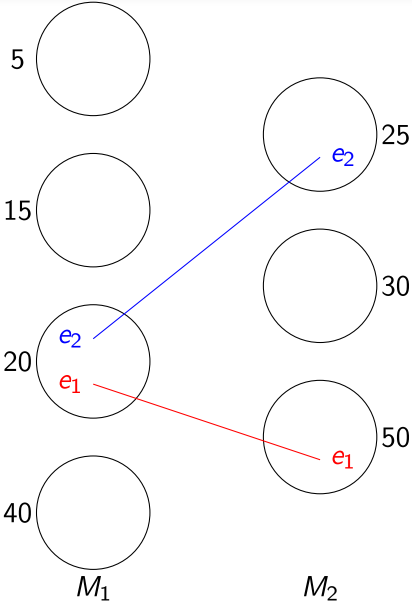
Recall: each max weight base has the same number of elements from any weight class.



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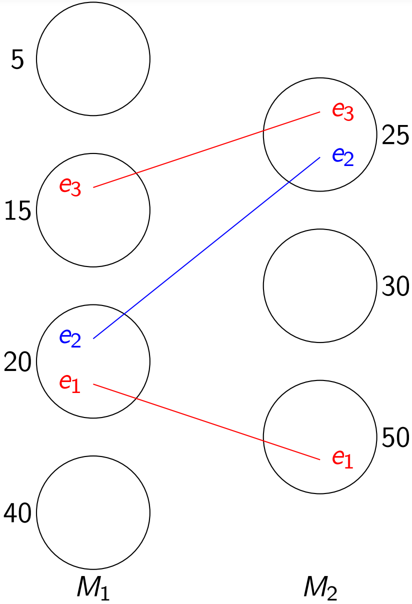

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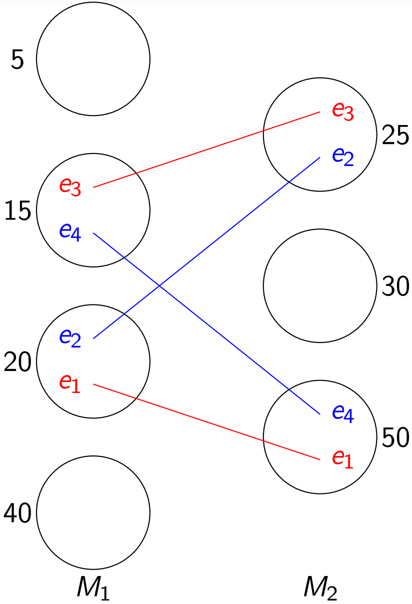

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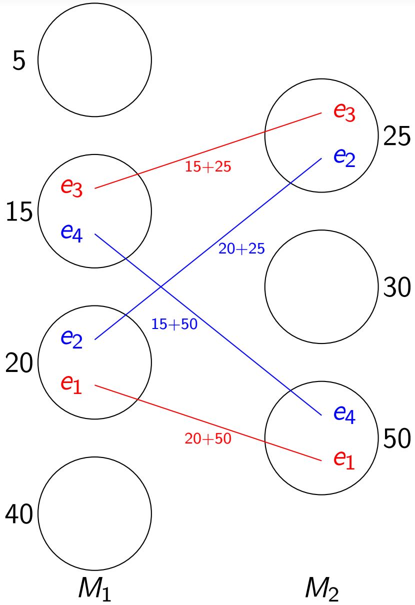

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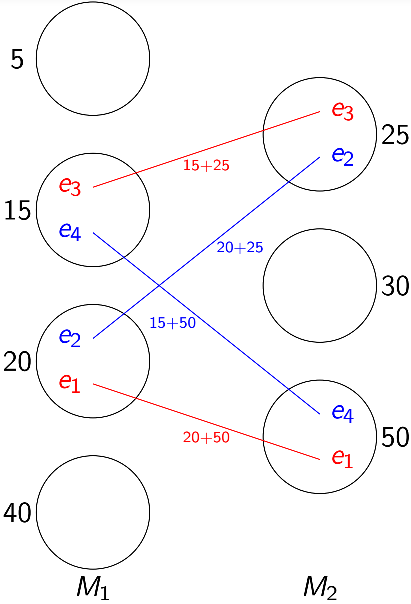

Figure: A cycle in two common bases

Obs. Alternating sum of the cycle

$$w(e_1)+w(e_3)-w(e_2)-w(e_4)$$

$$= 20 + 50 + (15 + 25)$$
$$-(20 + 25) - (15 + 50)$$
$$= 0.$$

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Efficiency:

- When there are no cycles of length $\leq 2^i$, the number of cycles of length $\leq 2^{i+1}$ is polynomial.

Questions

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- Search to decision reduction (in parallel) for bipartite matching.
- Search to weighted-decision: for what all optimization problems?

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