Strong Bounds for Arithmetic Progressions

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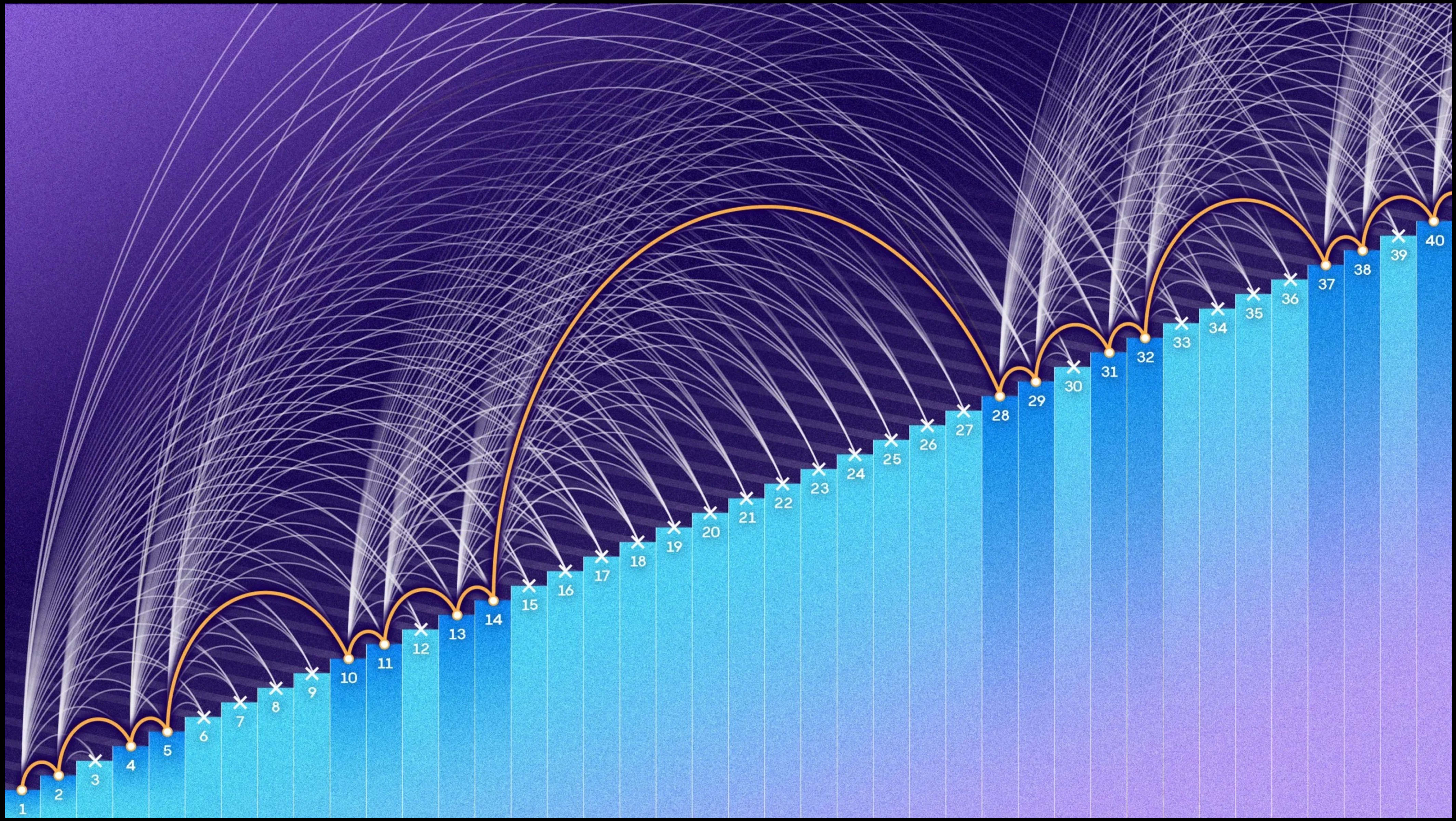
UIUC UCLA

3-Term Arithmetic Progressions

Erdos-Turan 36: How many numbers can we choose from {1,2,…,N} without three equally spaced numbers?



Picture from Quanta

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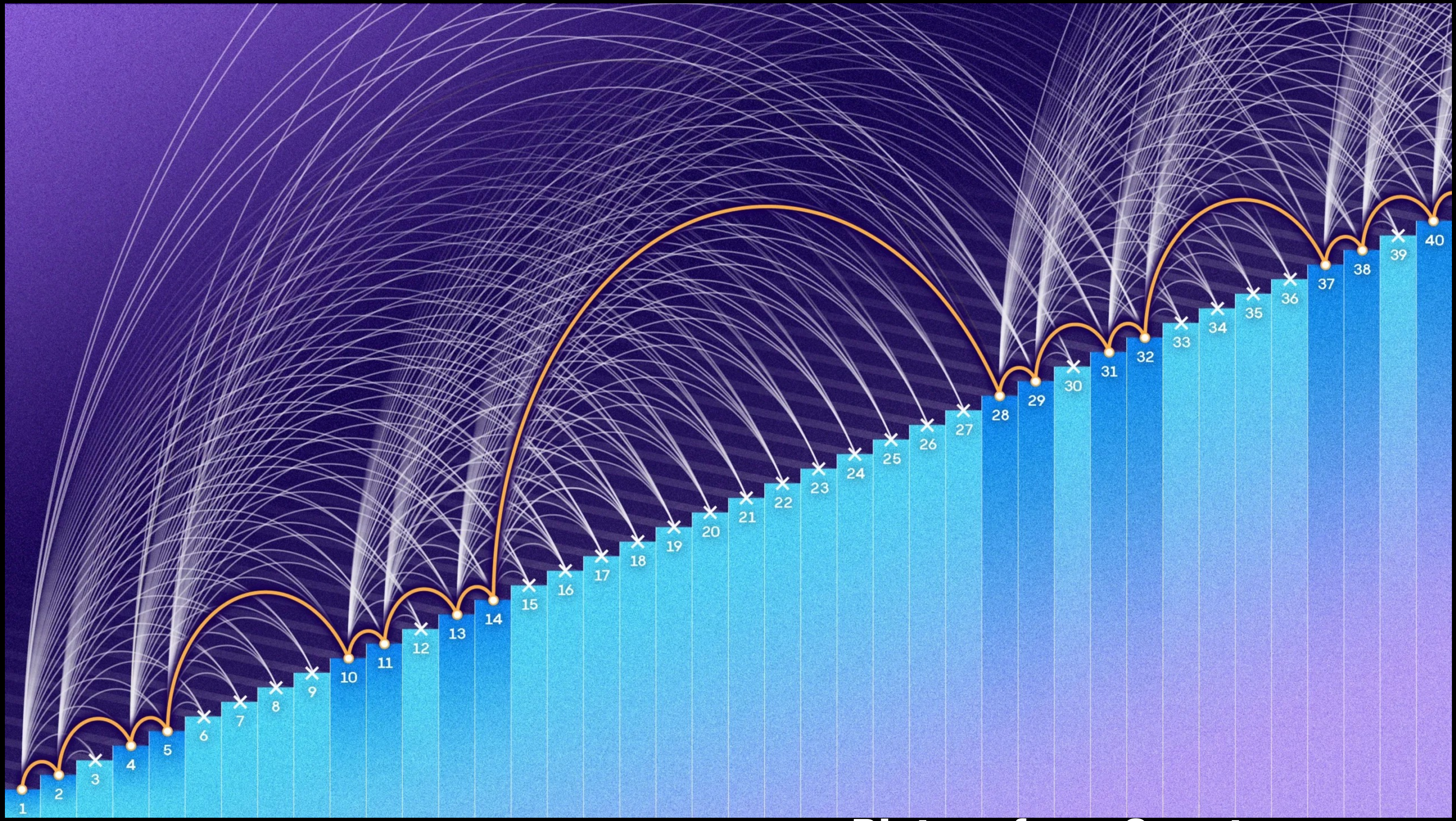

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1. Start with 1,2.
2. Skip 3.
3. Add 4,5.
4. Skip 6,7,8,9.
5. ….

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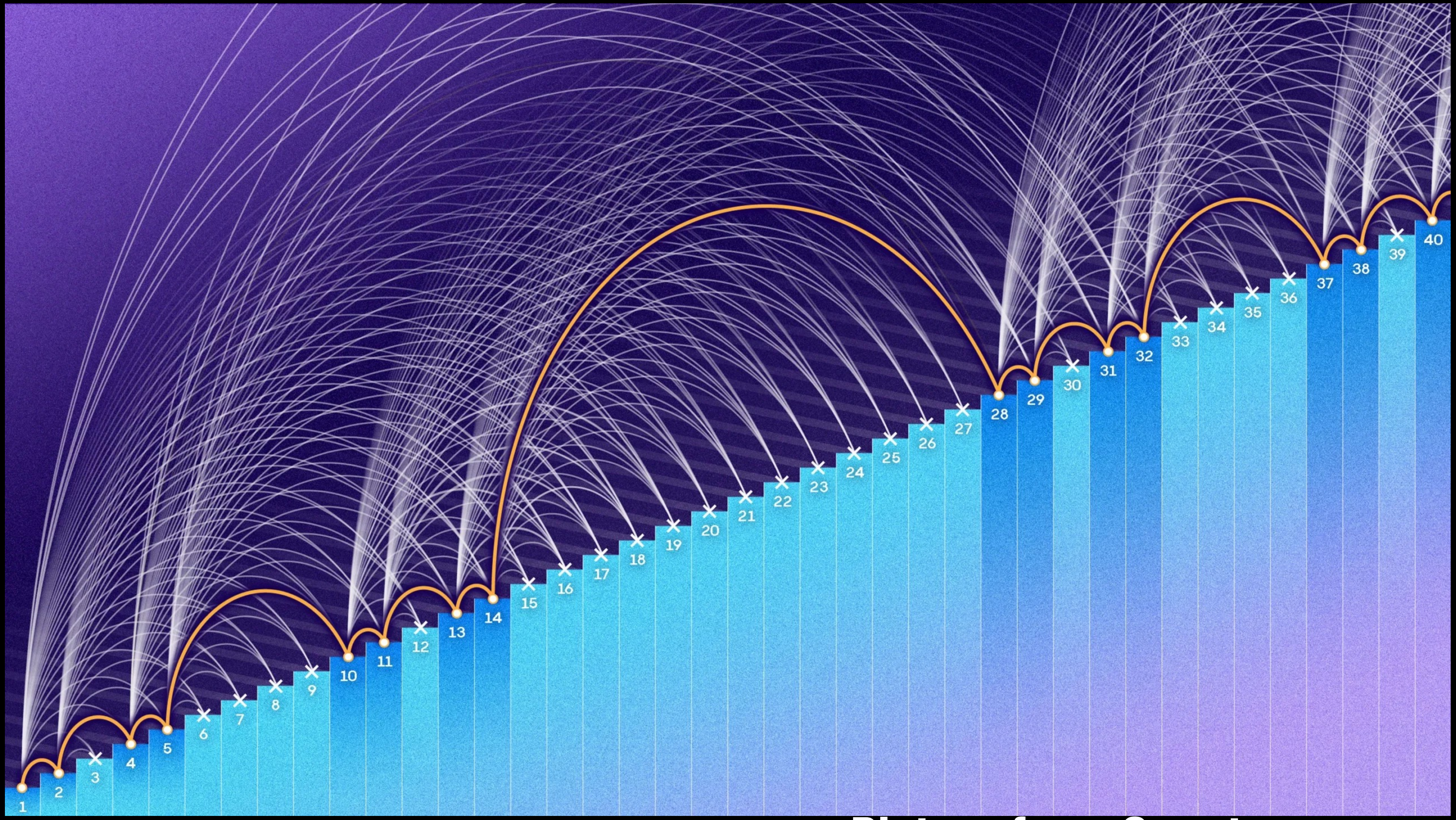

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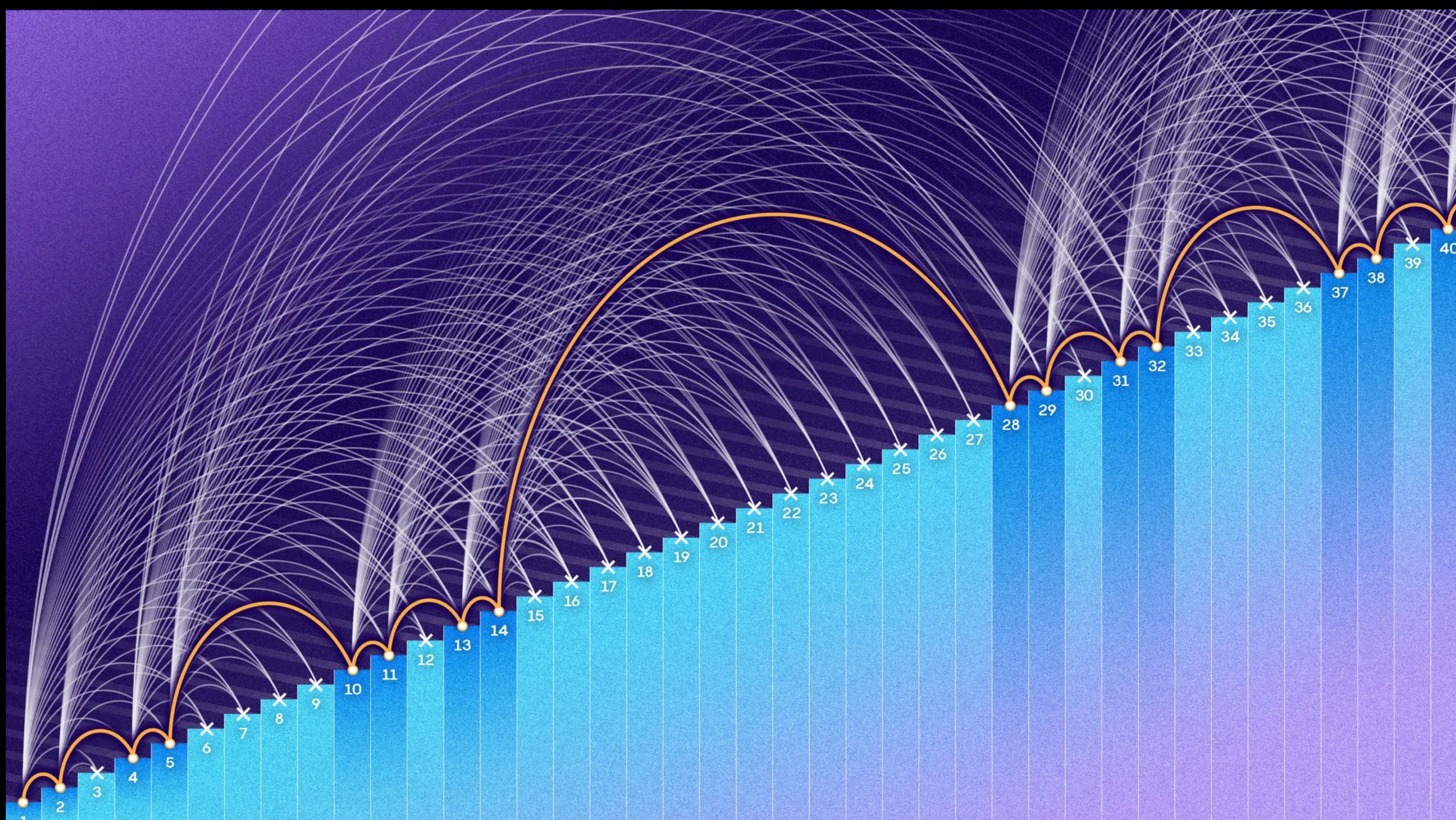

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Is this the best?

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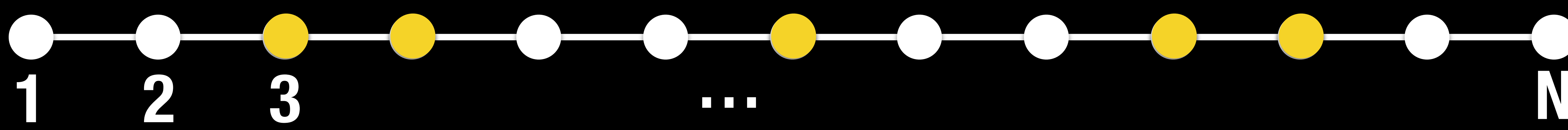

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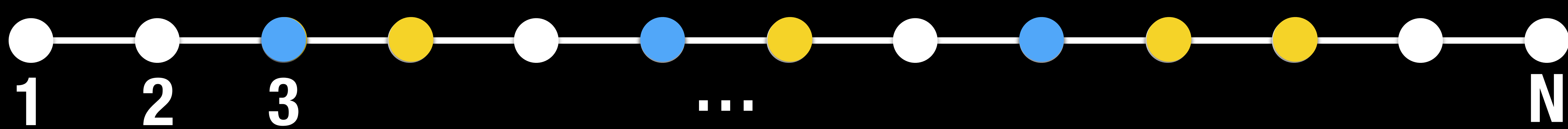

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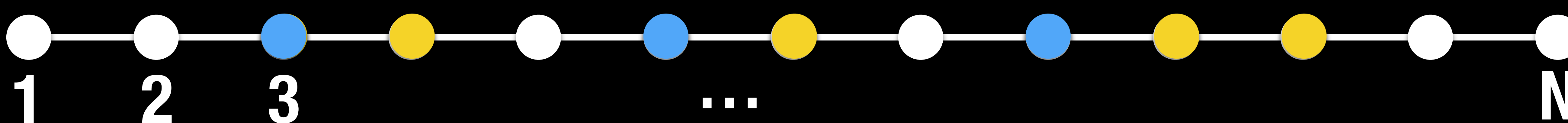

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Beautiful geometric construction. Application to communication complexity too!

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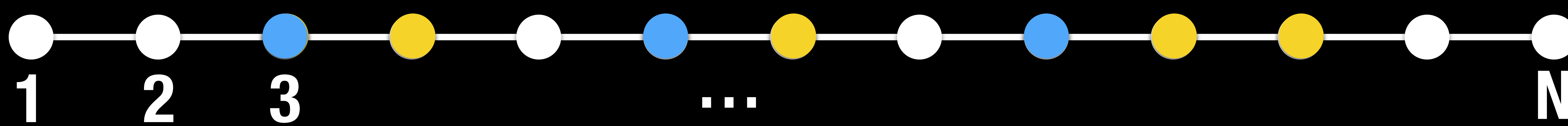

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Roth 53	$\delta \approx 1/(\log \log N)$
Heath-Brown, Szemeredi 87-90	$\delta \approx 1/(\log N)^c$, c **small**
Bourgain 99, 08	$\delta \approx 1/(\log N)^{1/2}, 1/(\log N)^{2/3}$
Sanders 11	$\delta \approx (\log \log N)^6/(\log N)$

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First progress on Erdos's problem on reciprocals …

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Number of triples is $2^{-O(\log(1/\delta)^{12})} N^2$.

A (Finite) Field Detour

Erdos-Turan 36: Subset A of integers {1,2,…,N}, $|A| = \delta N$. Want distinct $a, b, c \in A, a + b = 2c$.

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We have a subset A of $\{0,1,2\}^n$, $|A| = \delta 3^n$.

Are there unequal $a, b, c \in A$, $a + b = 2c \mod 3$?

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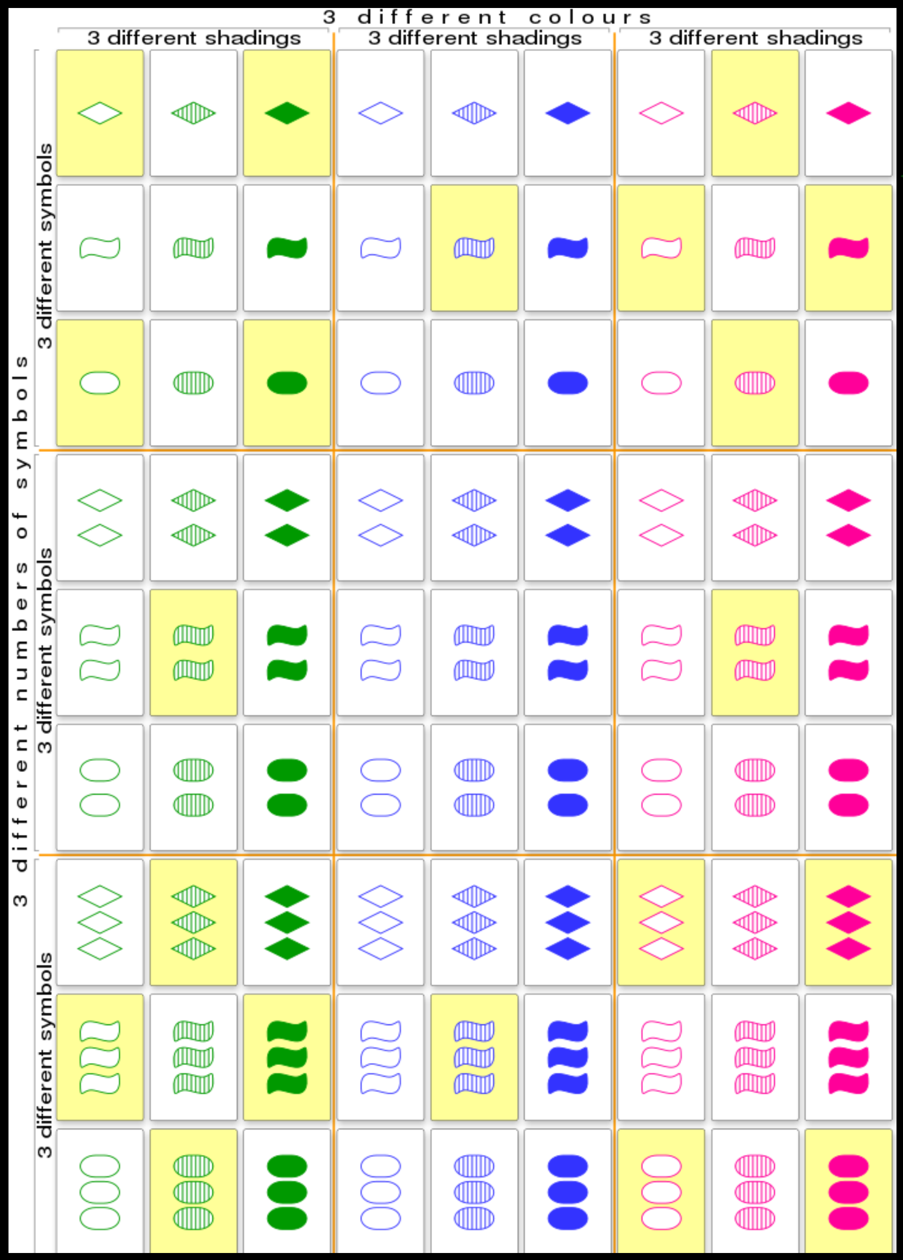

How many cards do you need in an n-dimensional SET game to find a SET?

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CLP17, EG17: $\delta \sim 3^{-cn}$ **suffices.**

"Cap-Set" problem.

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CLP17, E17: Proof "algebraic".

Doesn't quite carry to integers.

Roth, …,Bloom-Sisask 20: Proof "analytic".

We can "push" it to integers.

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This work: An analytic proof that $\delta \approx 2^{-\Omega((\log N)^{0.11})}$ suffices.

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Bloom-Sisask 23: A much cleaner way to push the finite field arguments to integers and new improvements.

Rest of the talk:

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1. Structure vs randomness

Density increment onto subspaces

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3AP problem has been the progenitor of many important techniques with wide applications …

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2. Spectral positivity

Underestimation to Overestimation

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1. Structure vs randomness	2. Spectral positivity	3. Sifting
Density increment onto subspaces	Underestimation to Overestimation	Finding structured substructures

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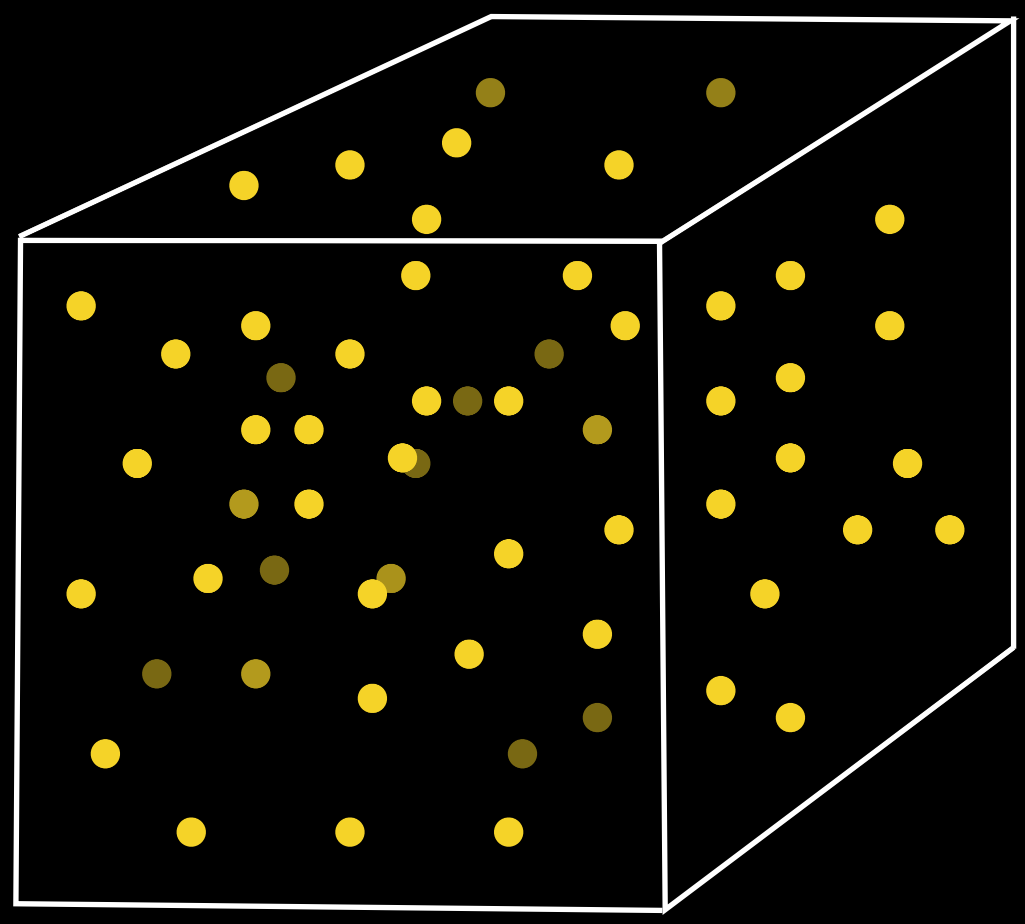
Structure vs Randomness



$A \subseteq \{0,1,2\}^n$

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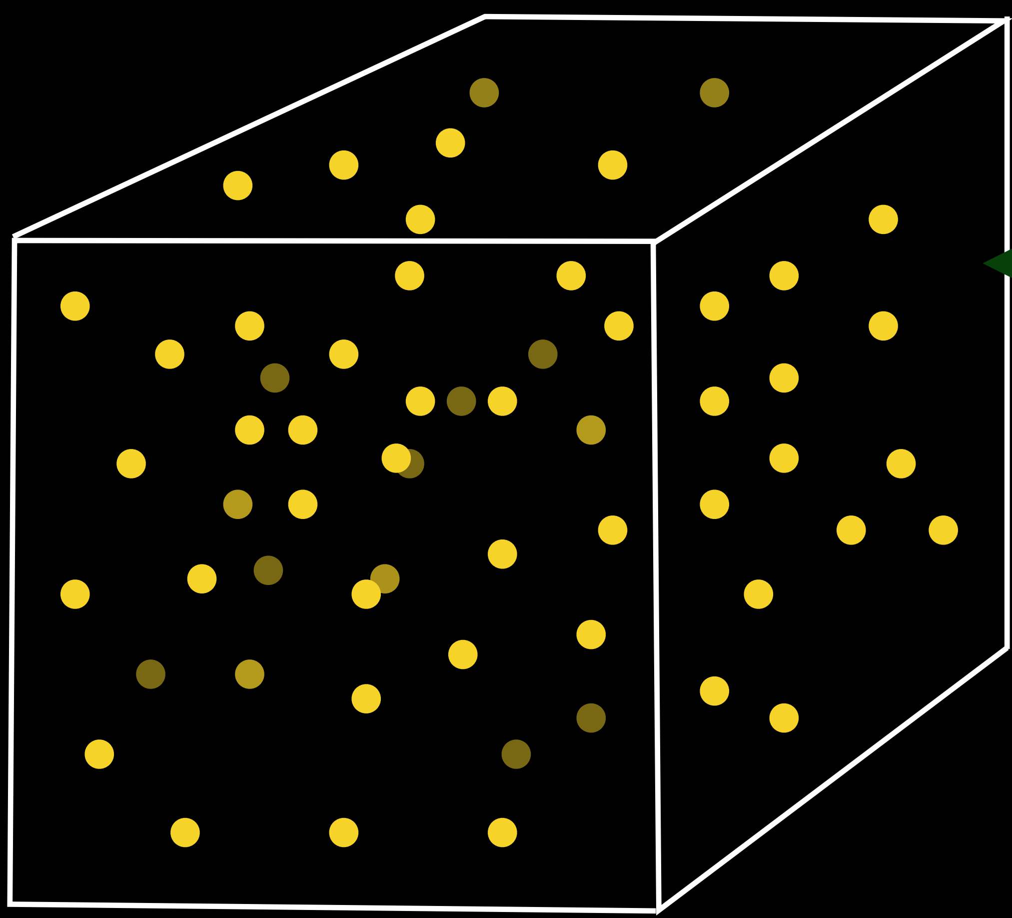

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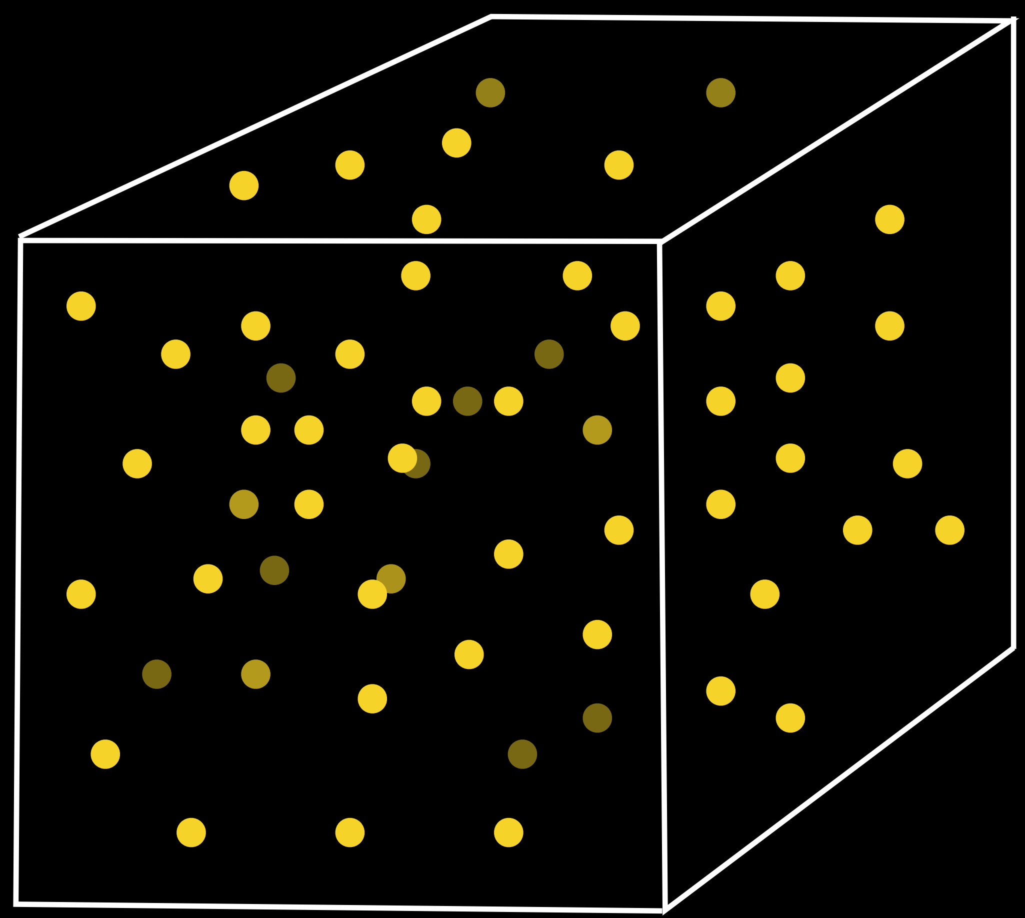
A is truly random and "density" δ:
E[Num. 3APs in A] $\approx \delta^3 N^2 \gg \delta N$.
Many non-trivial 3APs!

Structure vs Randomness

Wish 1: "Close to" uniform

$\Rightarrow$ **Many 3APs**



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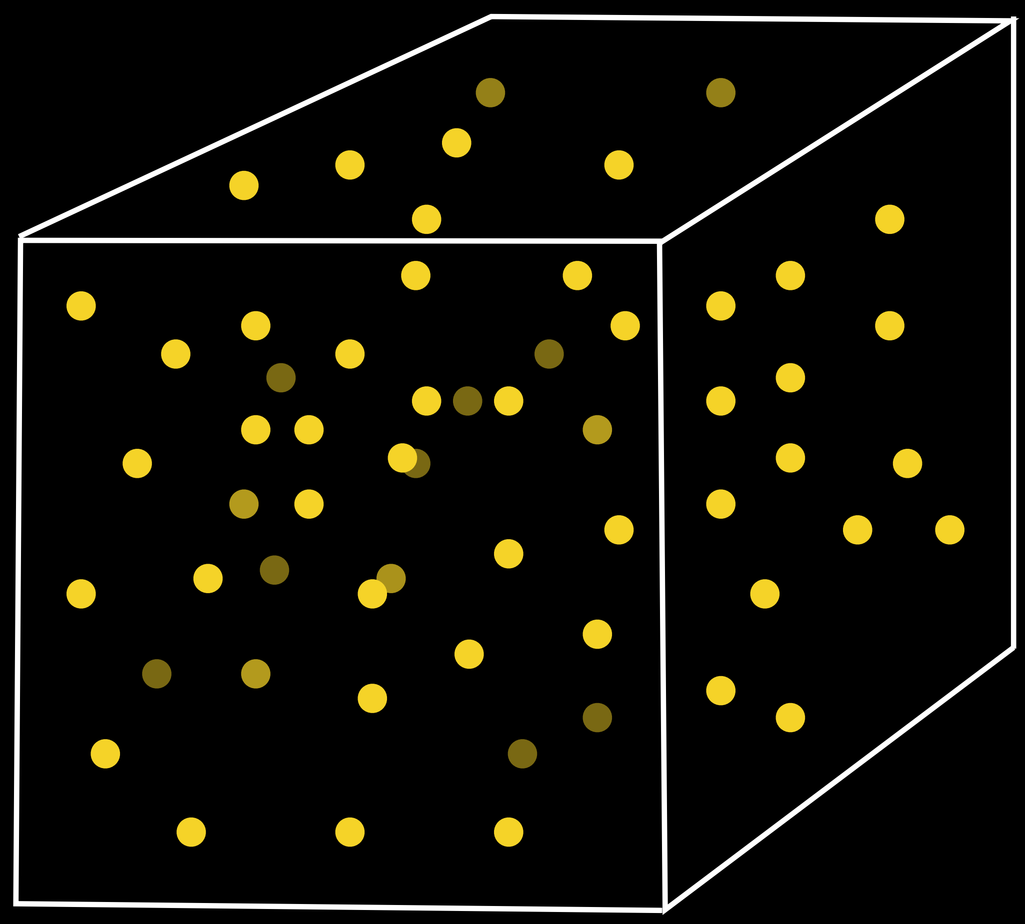

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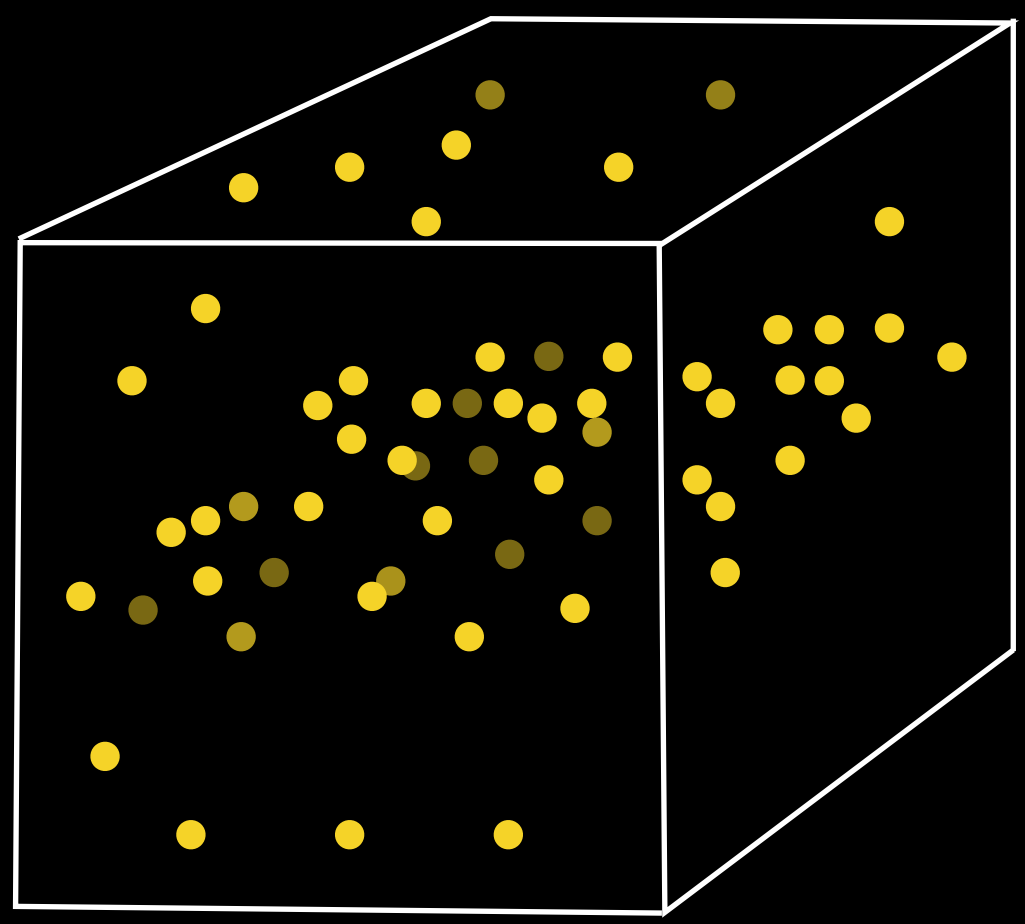

Structure vs Randomness

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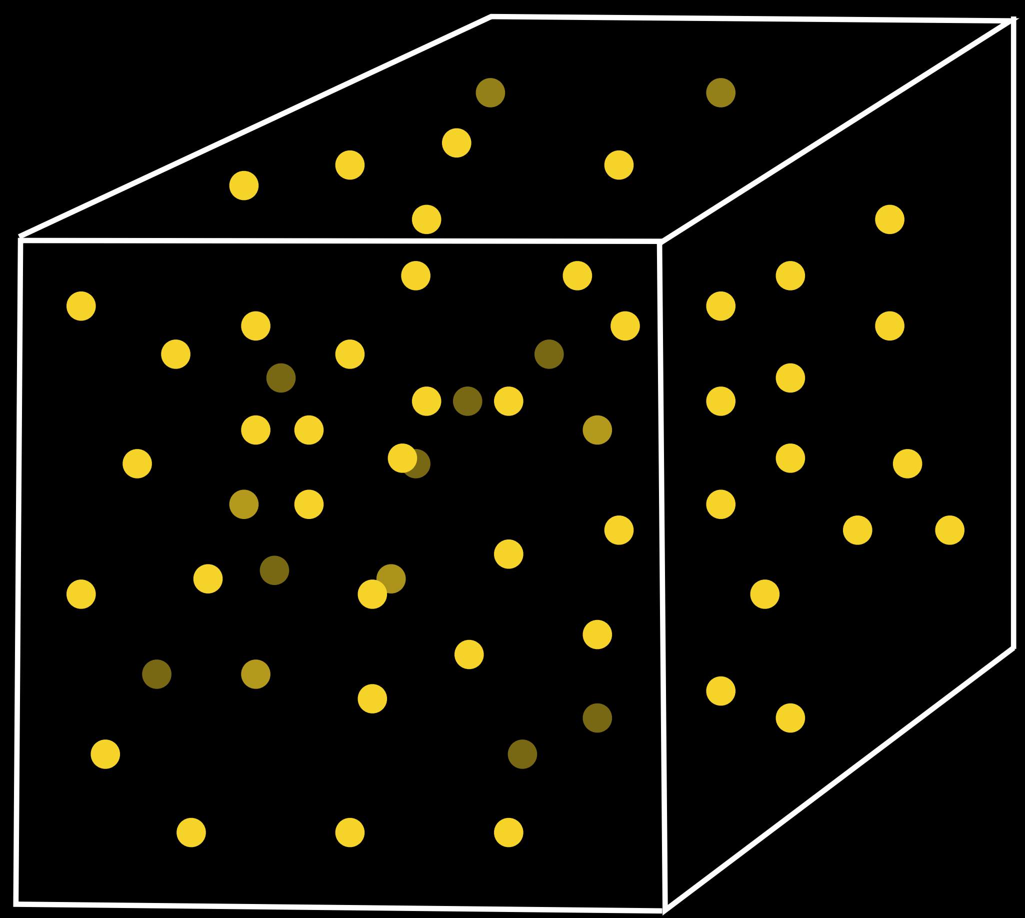

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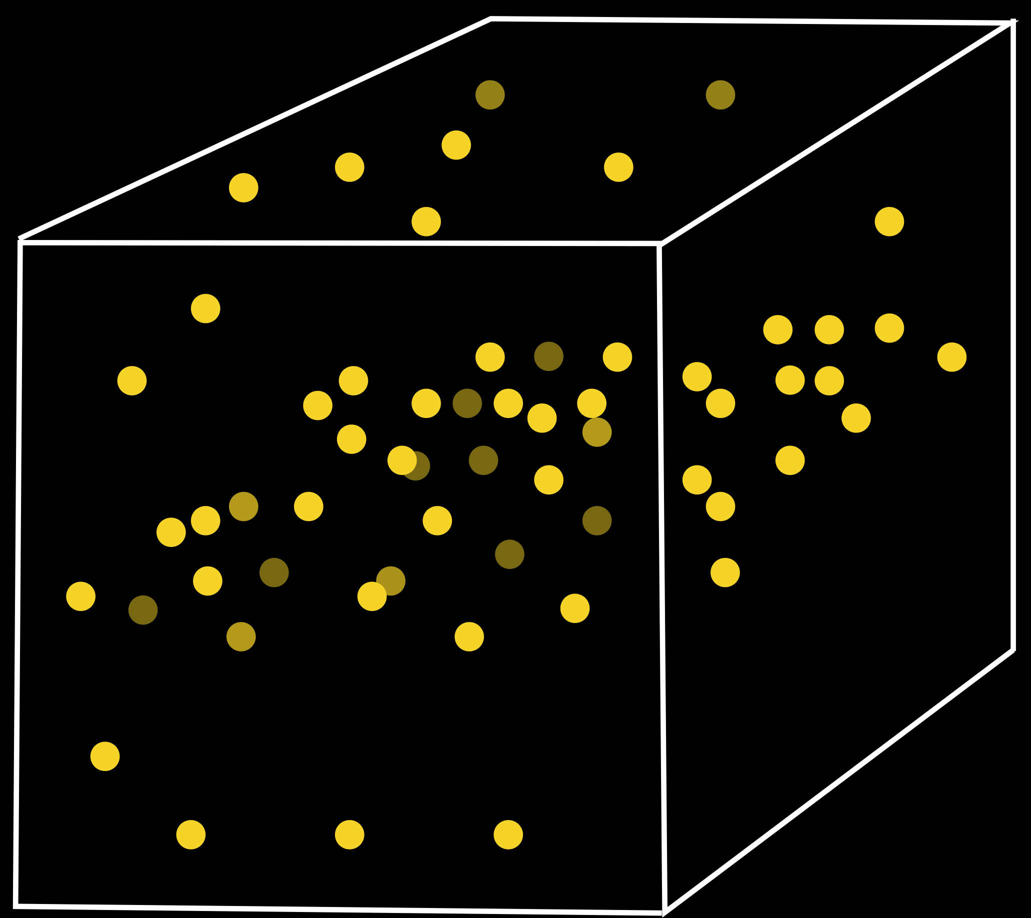
Wish 2: Far from uniform $\Rightarrow$ "Structure".



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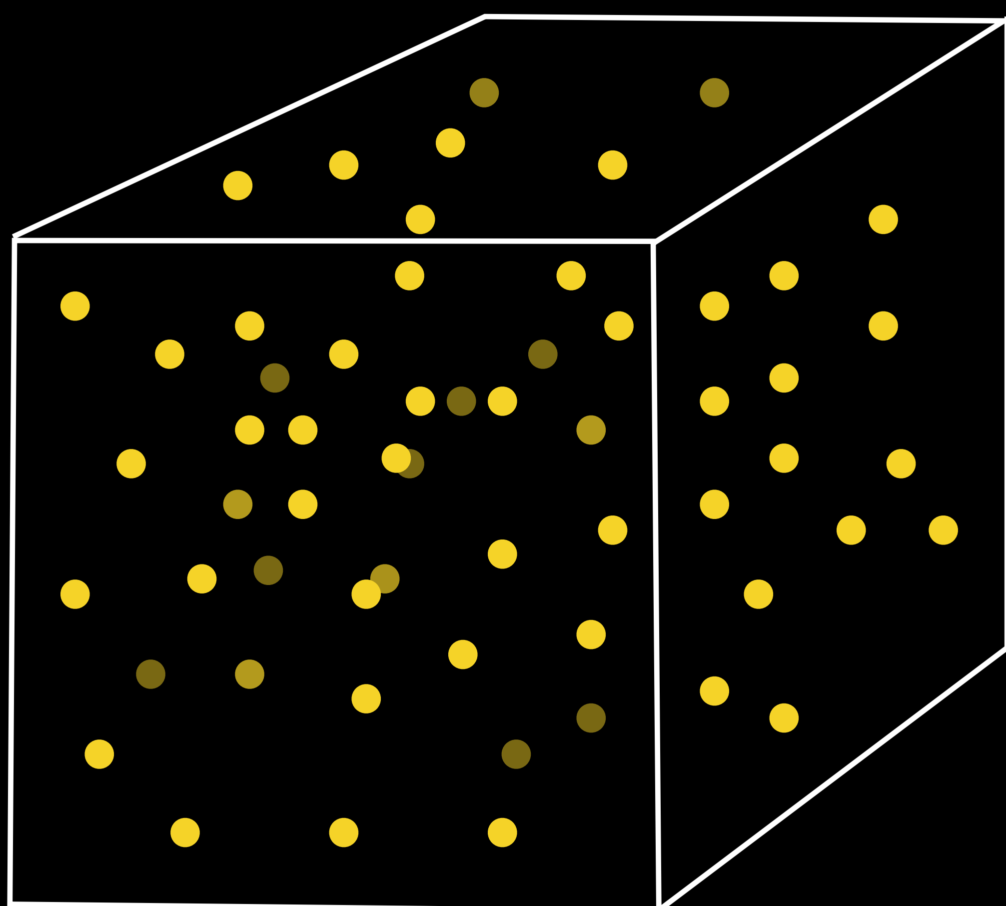

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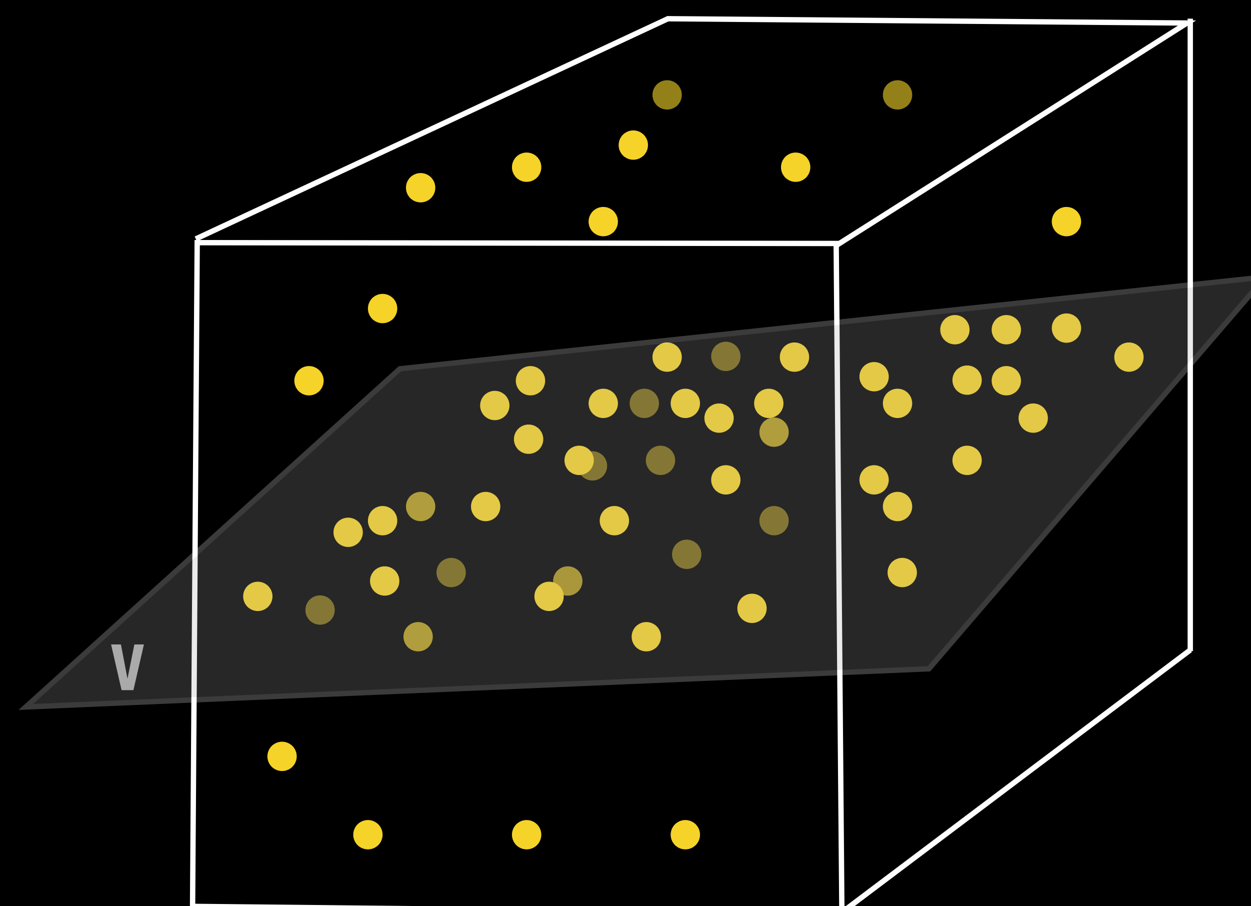

Affine space V

$|A \cap V| > \delta |V|$.

Induct on V

Structure vs Randomness

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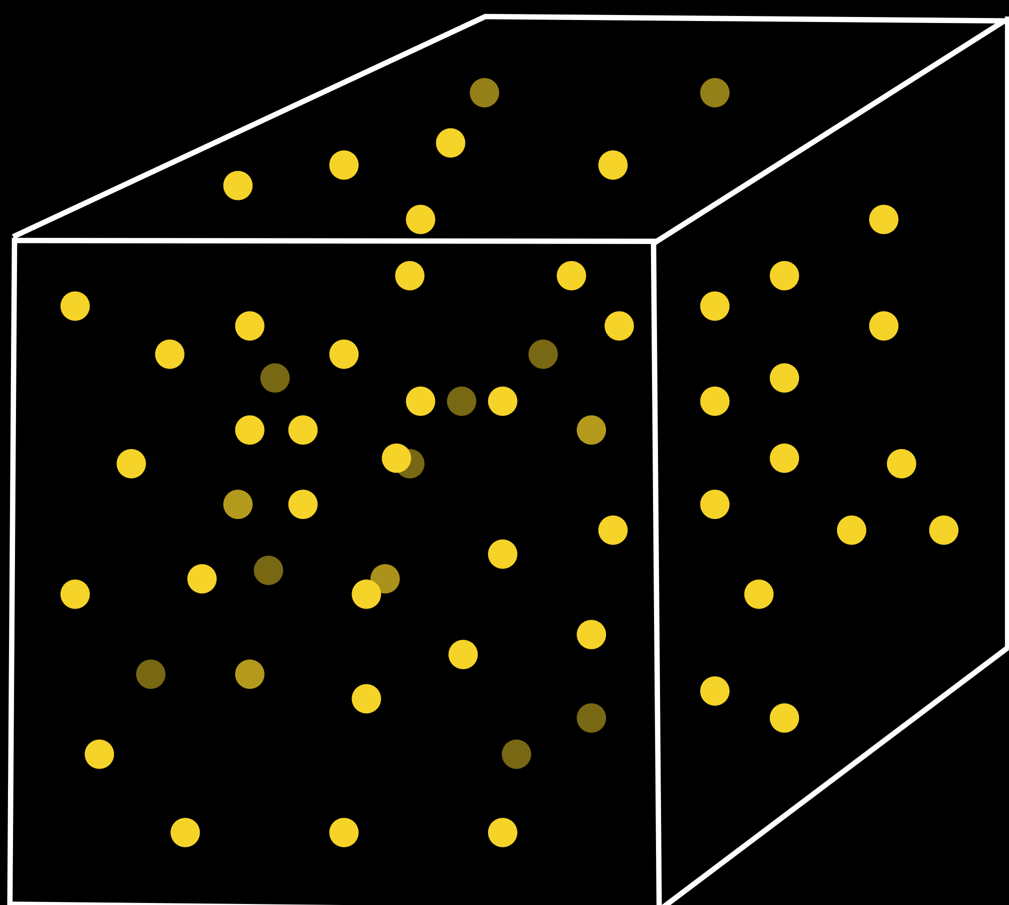
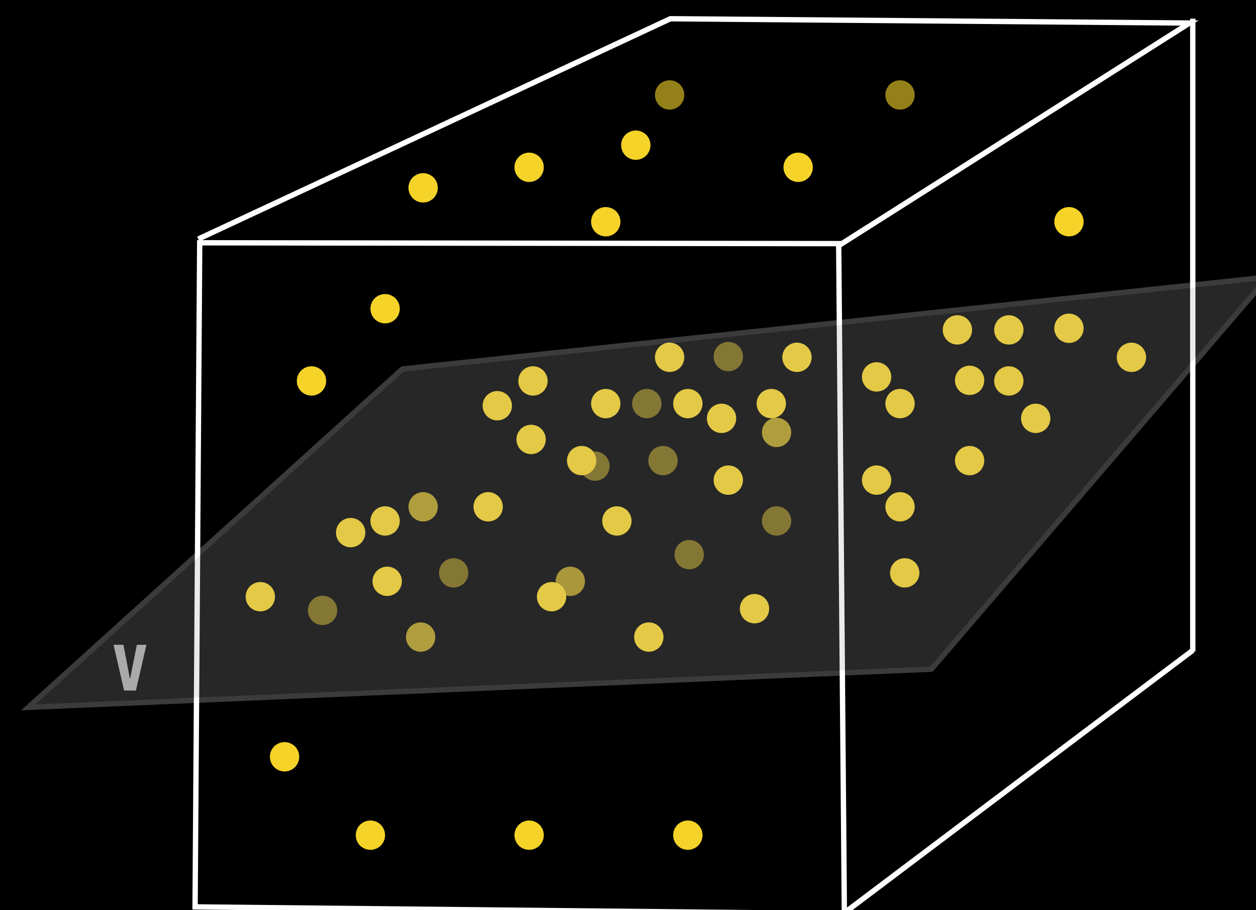
$\Rightarrow$ Many 3APs



$A \subseteq \{0,1,2\}^n$

$|A| = \delta N, N = 3^n$.

Wish 2: Far from uniform $\Rightarrow$ affine space V with density increment , co-dim(V) small.



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Roth-Meshulam Proof

Wish 1: “Close to” uniform
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$A \subseteq \mathbb{F}_3^n$
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Roth-Meshulam Proof

Wish 1: "Close to" uniform
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1. A is small-biased $\Rightarrow$ Many 3APs.

Thm: Uniform on A fools linear tests with error $\delta/2$, then number of 3APs in A $> \delta^3 N^2/2$.

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$$Pr[\langle w, X\rangle = \alpha] = \frac{1}{3} \pm \frac{\delta}{6}.$$

$X \sim A, \forall w \in \{0,1,2\}^n, \alpha \in \{0,1,2\}$.

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2. A not small-biased $\Rightarrow$ density inc.

Thm: If A not small-biased $\Rightarrow$ affine space V of co-dimension 1, $|A \cap V| > (\delta + \Omega(\delta^2))|V|$.

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$A, \mathbb{F}_3^n$, **density** δ

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$A, \mathbb{F}_3^n$, **density** δ → (**No 3AP**) → $A_1 = A \cap V_1, V_1$, **density** $\delta + \delta^2$

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$A_2 = A_1 \cap V_2, V_2$, **density** $\delta + 2\delta^2$

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$A_2 = A_1 \cap V_2, V_2$, **density** $\delta + 2\delta^2$

← $1/\delta$ **Iterations** ←

$A_l = A \cap V_l, V_l$, **density** $1.01 \cdot \delta$

$A \subseteq \mathbb{F}_3^n$
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→

Contradiction: $\delta > 100/n$.

$A \subseteq \mathbb{F}_3^n$

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Roth-Meshulam Proof

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←

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Thm: $A \subseteq \mathbb{F}_3^n, |A| \gg 3^n/n,$ **then A has many 3APs.**

$A \subseteq \mathbb{F}_3^n$

$|A| = \delta N, N = 3^n$.

Going beyond?

Wish 1: "Close to" uniform $\Rightarrow$ Many 3APs

Wish 2: Far from uniform $\Rightarrow$ affine space V with density increment , co-dim(V) small.

$A \subseteq \{0,1,2\}^n$
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Step 1. "Close" to uniform $\Rightarrow$ Many 3APs?

Step 2. Far from uniform $\Rightarrow$ affine V with (1.01) density increment, co-dim$(V) \leq poly(\log(1/\delta))$?

Previous arguments:
Need co-dim$(V) \approx 1/\delta$.

$A \subseteq \{0,1,2\}^n$
$|A| = \delta N, N = 3^n$.

Going beyond?

Step 1. "Close" to uniform $\Rightarrow$ Many 3APs?

Let $2A = \{2c : c \in A\}$.

Want: Many $a, b \in A$ **such that** $a + b \in 2A$.

Recall: Want distinct $a, b, c \in A, a + b = 2c$

$A \subseteq \{0,1,2\}^n$

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Want: Many $a, b \in A$ **such that** $a + b \in 2A$.

Quantitatively?

$A \subseteq \{0,1,2\}^n$

$|A| = \delta N, N = 3^n$.

Going beyond?

Step 1. "Close" to uniform $\Rightarrow$ Many 3APs?

Let $2A = \{2c : c \in A\}$.

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X,Y independent, uniform over A

$$Pr[X + Y \in 2A] = \Omega(\delta)?$$

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Distribution of X+Y?

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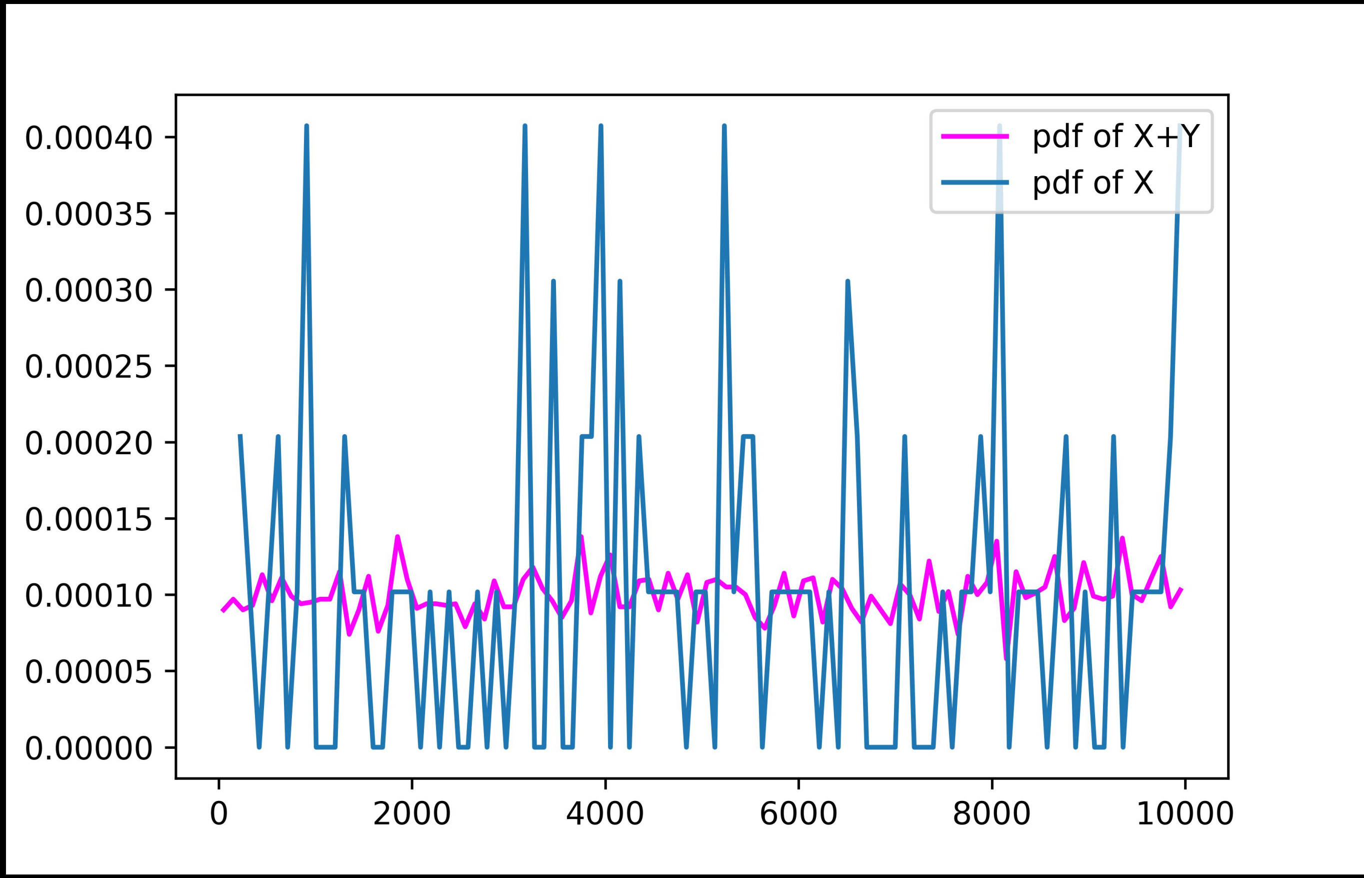

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A Star Arises

Step 1. "Close" to uniform $\Rightarrow$ Many 3APs?

X,Y independent, uniform over A

$$Pr[X + Y \in 2A] = \Omega(\delta)?$$

Fact: If density of X is μ_A,
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Convolutions: $f * g(x) = E_z[f(z)\, g(x - z)]\,.$

$f \circ g(x) = E_z[f(z)\, g(x + z)].$

$A \subseteq \{0,1,2\}^n$
$|A| = \delta N, N = 3^n\,.$

Closeness: Lowerconcentration

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for every $T, |T| = \delta N$, $Pr[X + Y \in T] = \Omega(\delta)$?

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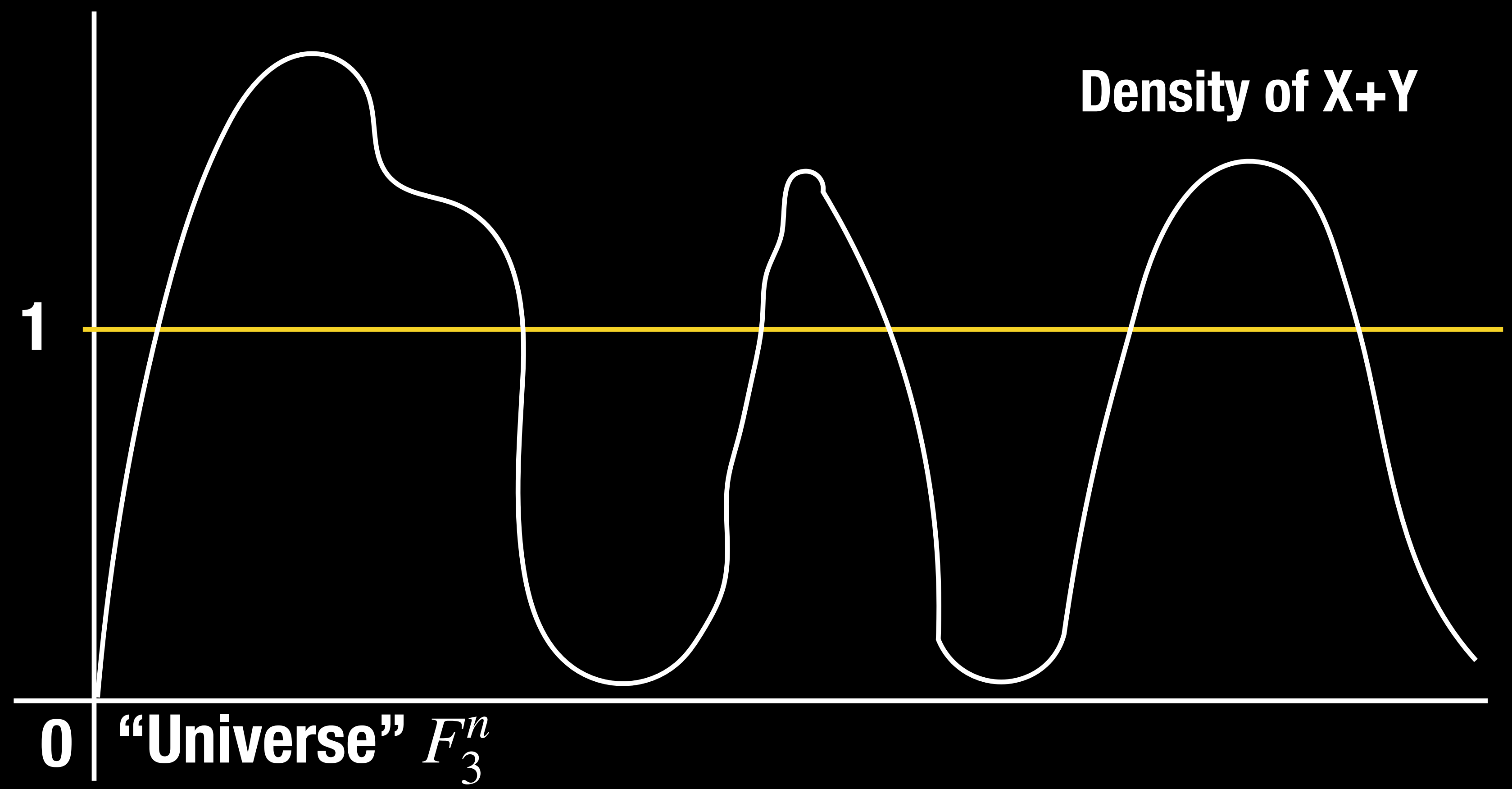

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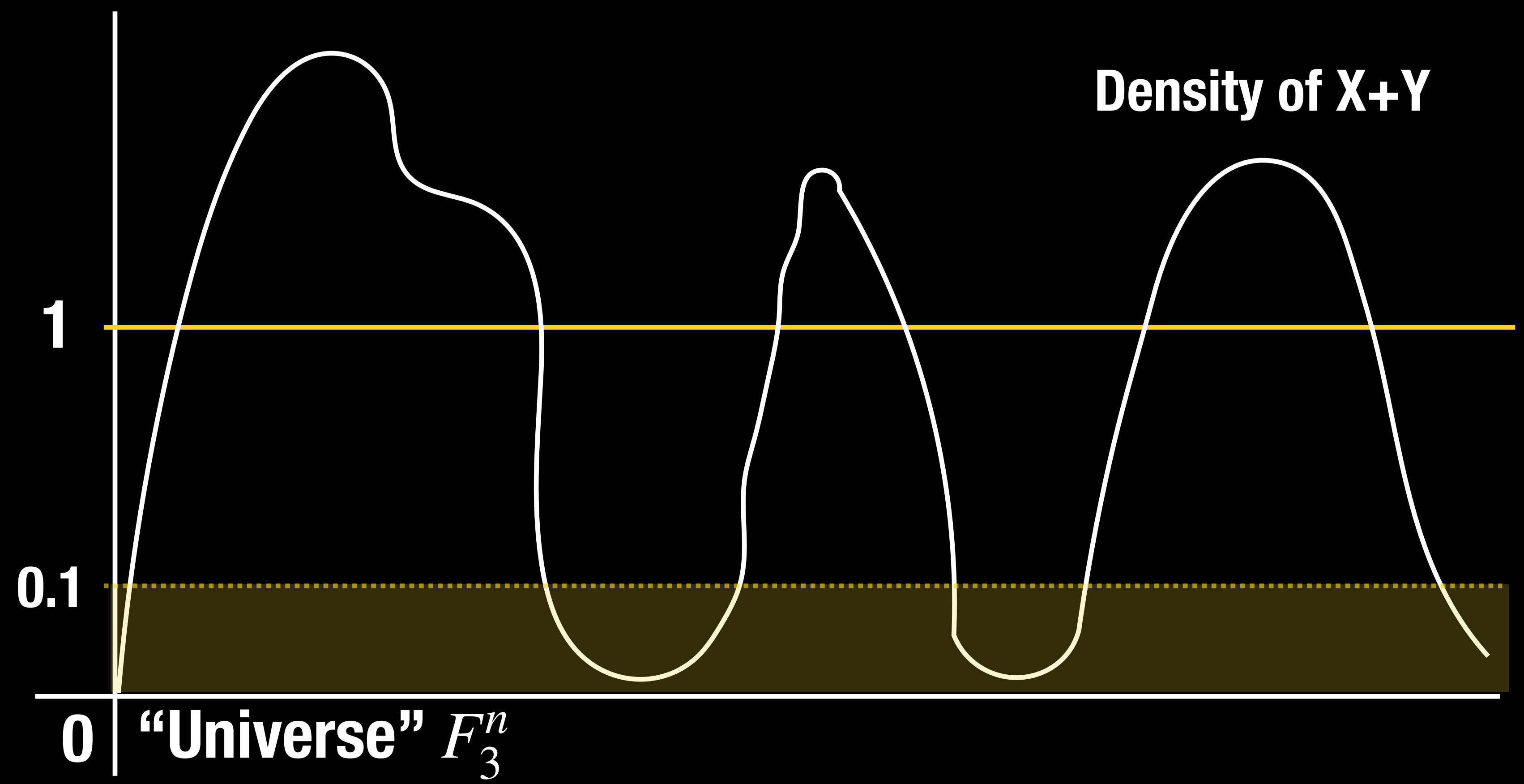

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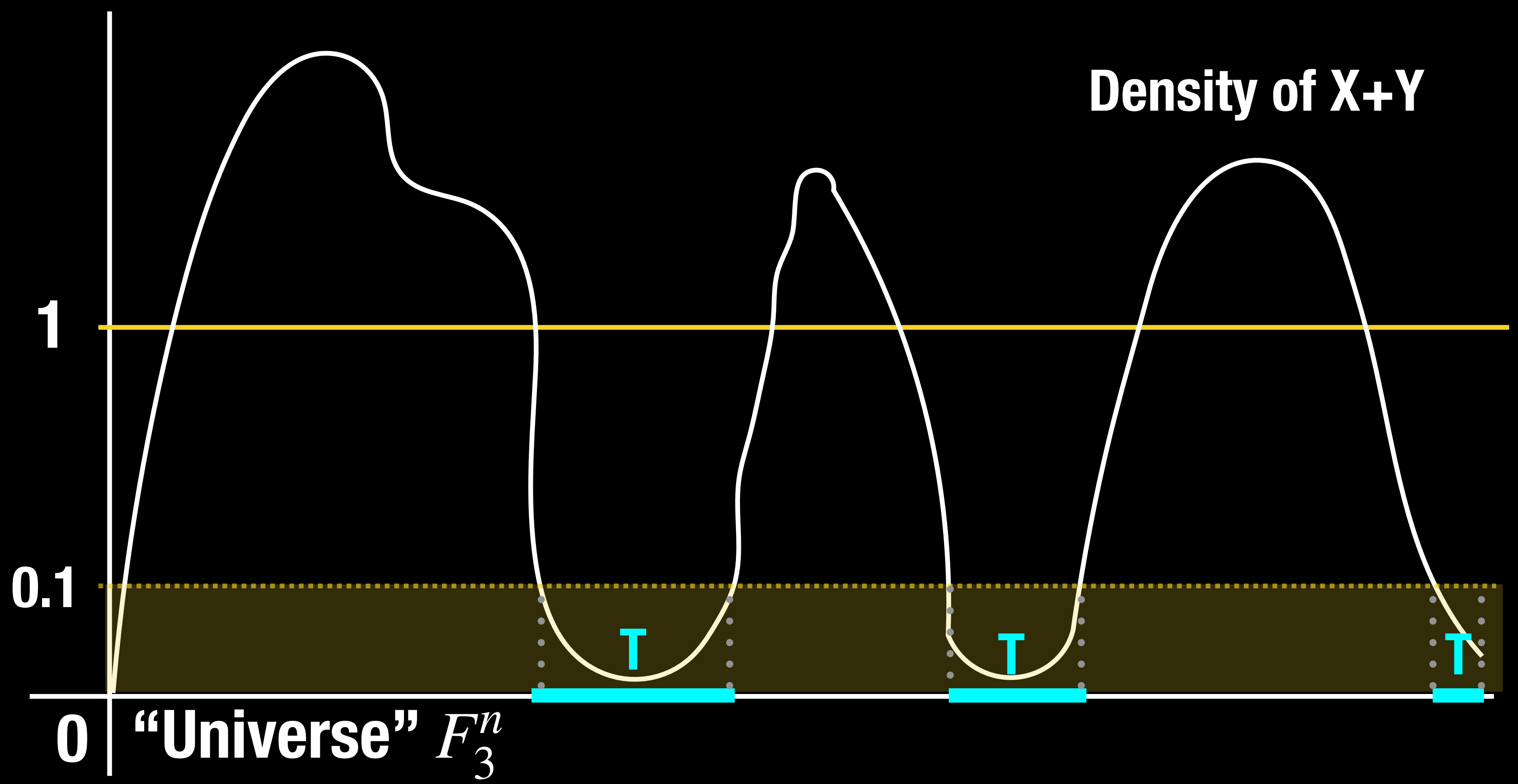

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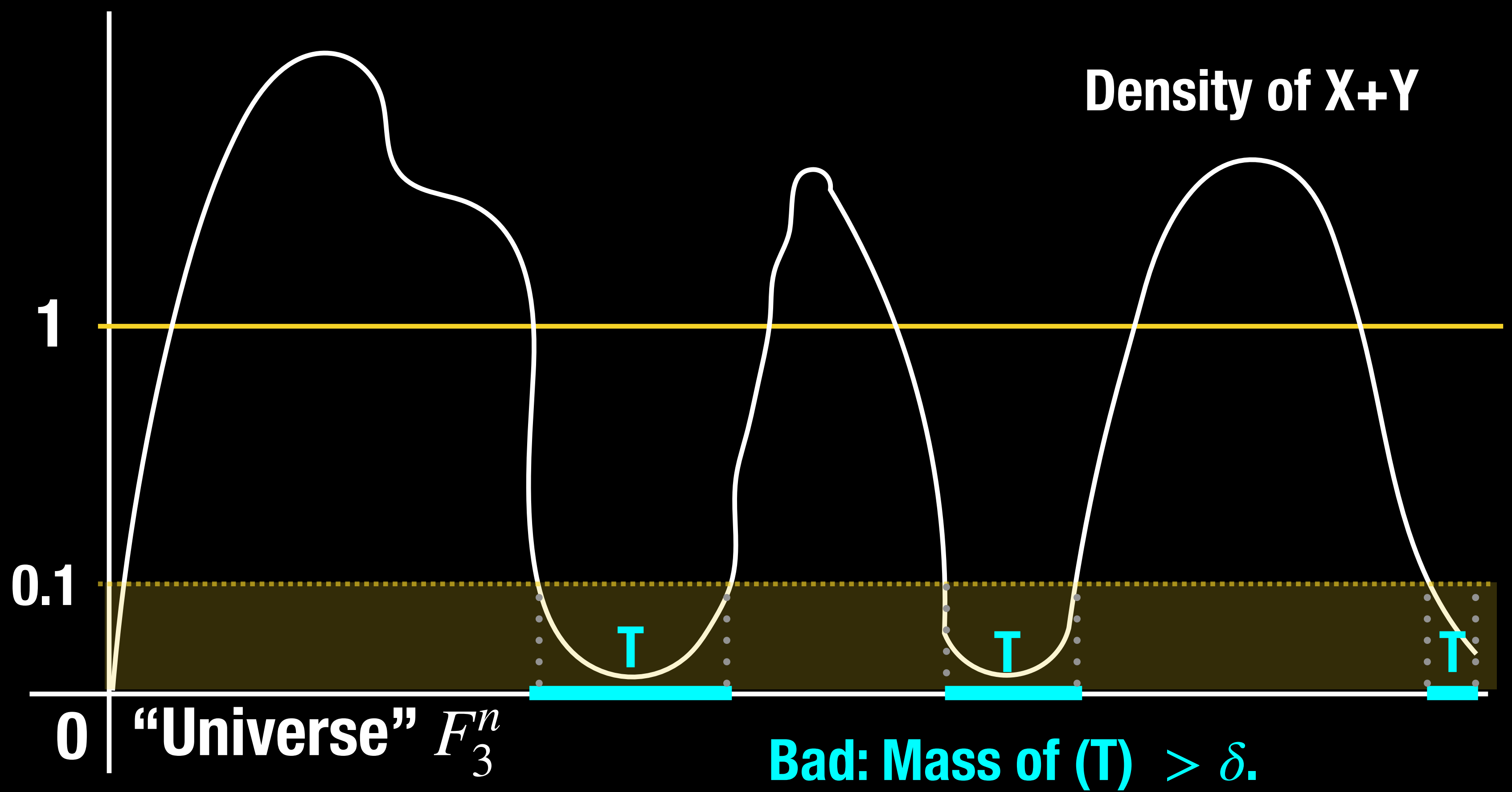

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Closeness: Lowerconcentration

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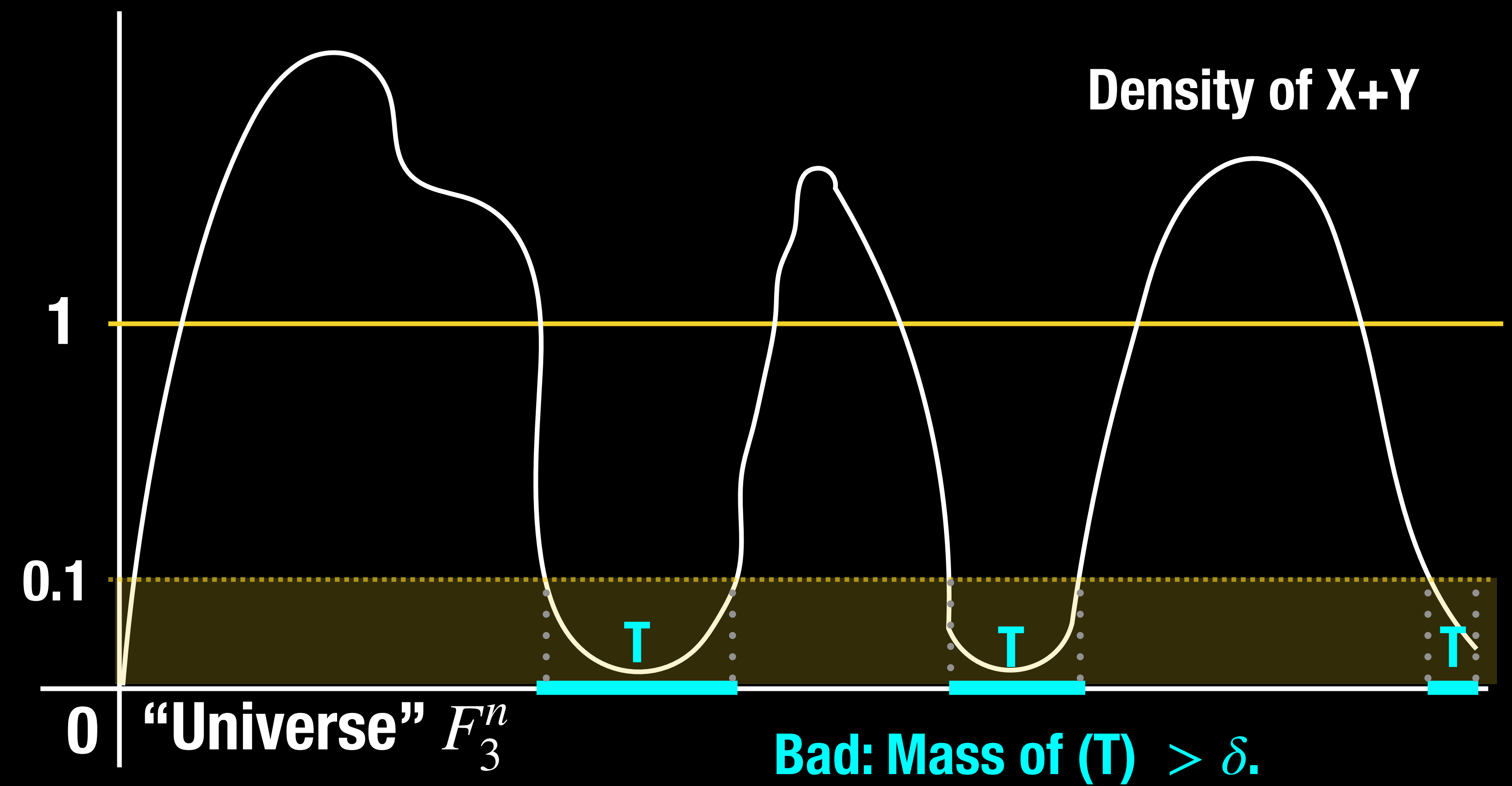
Def: Density q δ-lower-concentrated if

$$Pr[q(z) < 0.1] < \delta/2.$$



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S vs R Lemma: $\mu_A * \mu_A$ lower-concentrated, then we have many 3APs.

Proof: $Pr[X + Y \in 2A] \geq \delta/20.$
Number of 3APs is $\approx \delta^3 N^2/20.$

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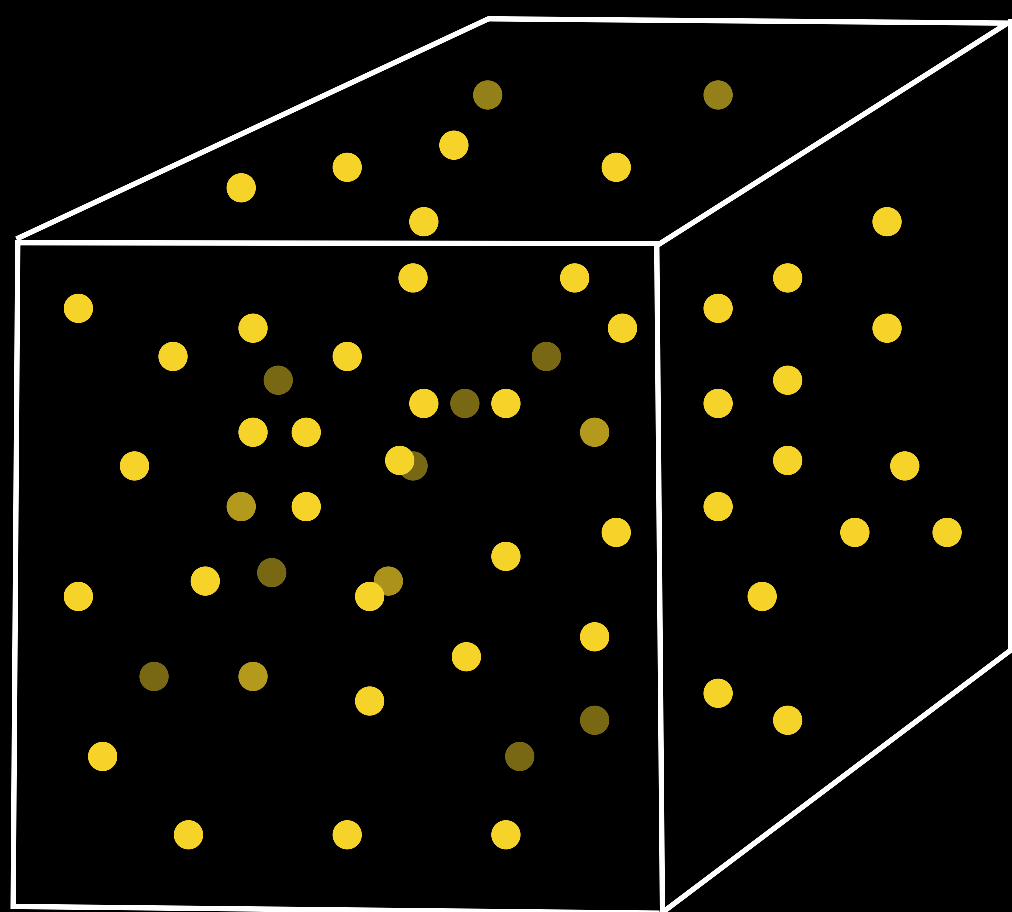
Structure vs Randomness: Summary

Wish 1: "Close to" uniform
$\Rightarrow$ Many 3APs

Wish 2: Far from uniform $\Rightarrow$ affine space V with density increment , co-dim(V) small.



$A \subseteq \mathbb{F}_3^n$

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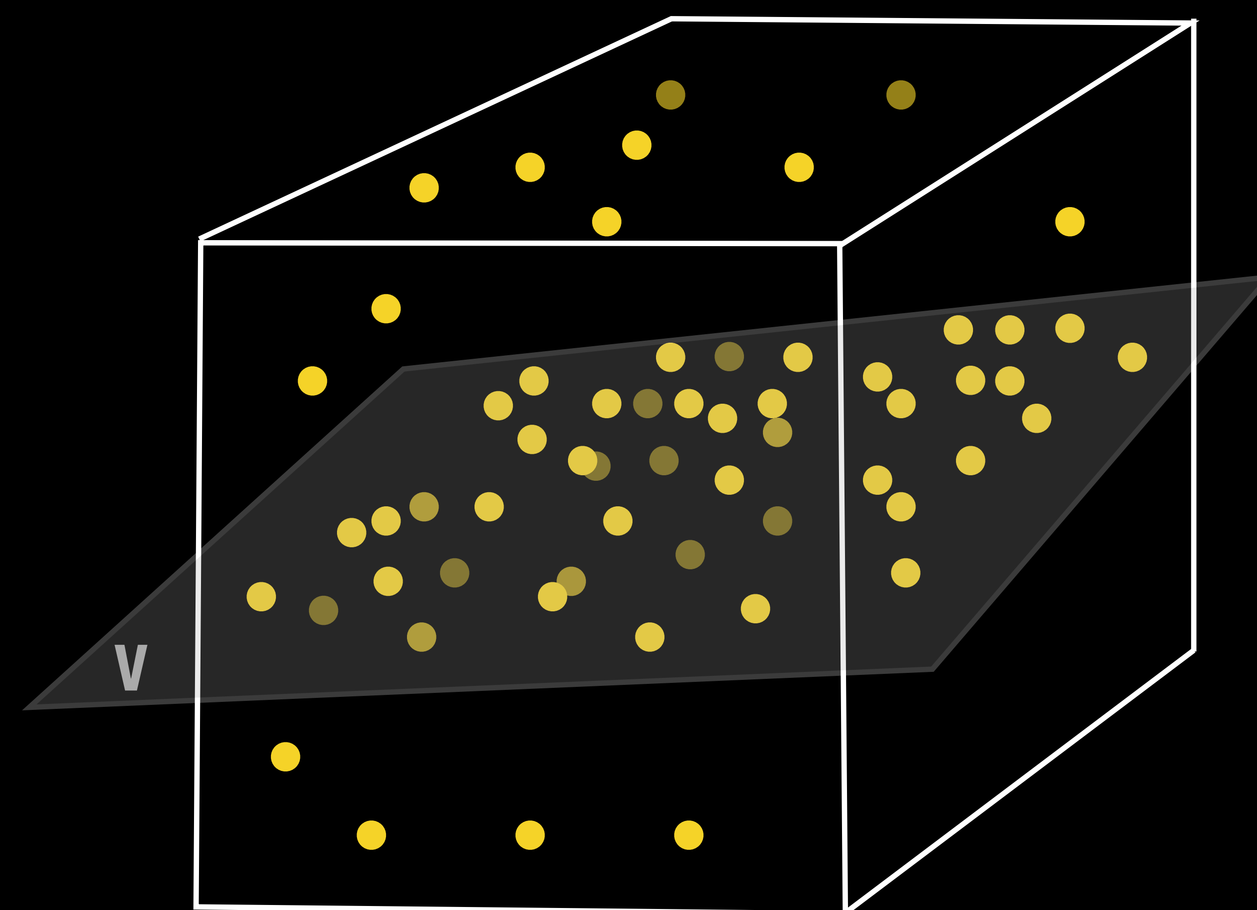

Affine space V

$|A \cap V| > \delta |V|.$

Structure vs Randomness Summary

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Need: $\mu_A \circ \mu_A$ **not lower-concentrated** $\Rightarrow$
(Strong) Density increment

$A \subseteq \{0,1,2\}^n$
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Overview

We have a subset A of F_q^n, $|A| = \delta q^n$.

How small can δ be, $\exists$ unequal $a, b, c \in A$, $a + b = 2c$?

An analytic proof that $\delta \sim 2^{-\Omega((\log N)^{0.11})}$ suffices.

1. Density increment

Structure vs pseudorandomness

2. Spectral positivity

Underestimation to Overestimation

3. Sifting

Finding structured substructures

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1. Density increment	2. Spectral positivity	3. Sifting
Structure vs pseudorandomness	Underestimation to Overestimation	Finding structured substructures

Spectral Positivity

Want: $\mu_A \circ \mu_A$ **not lower-concentrated**

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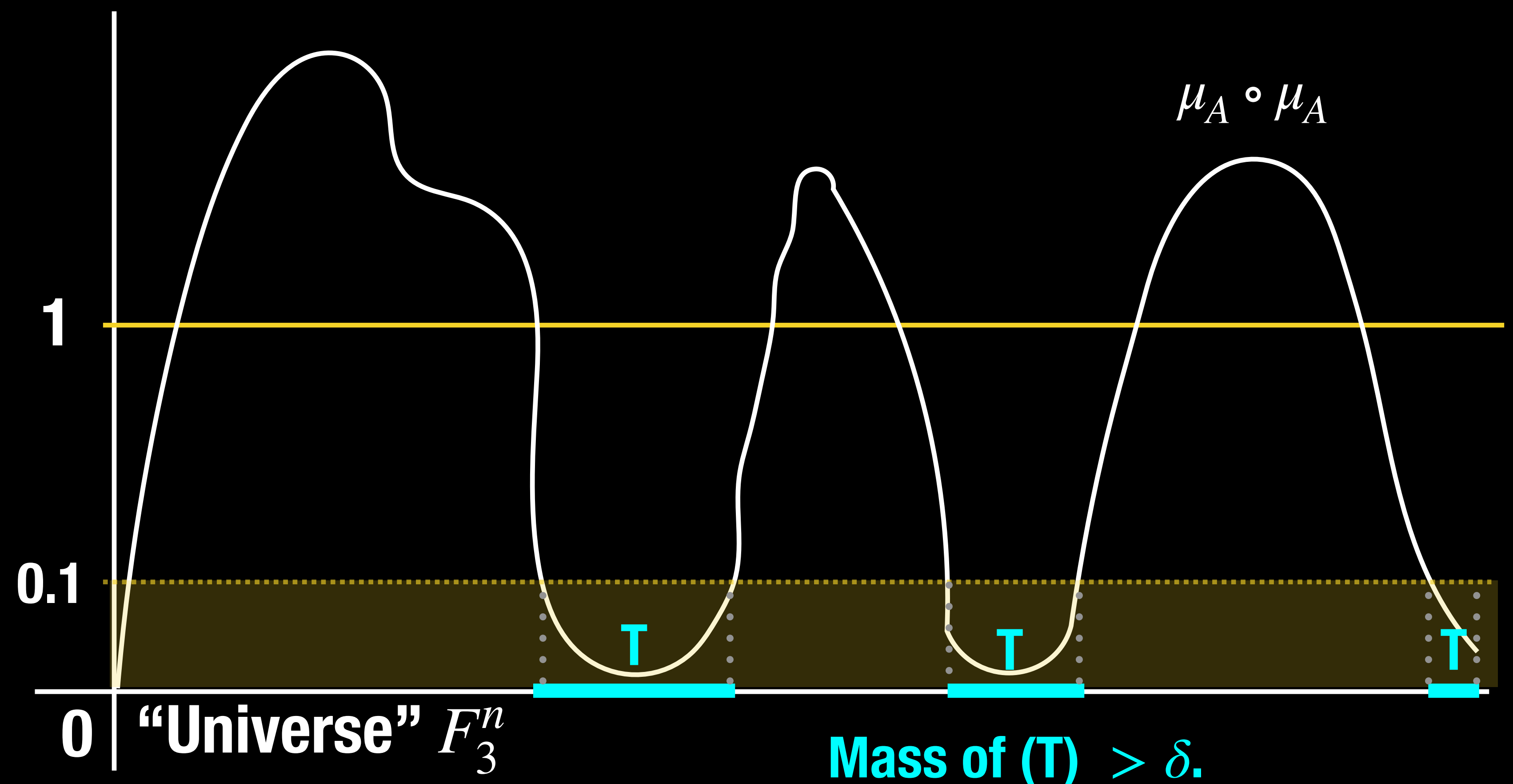

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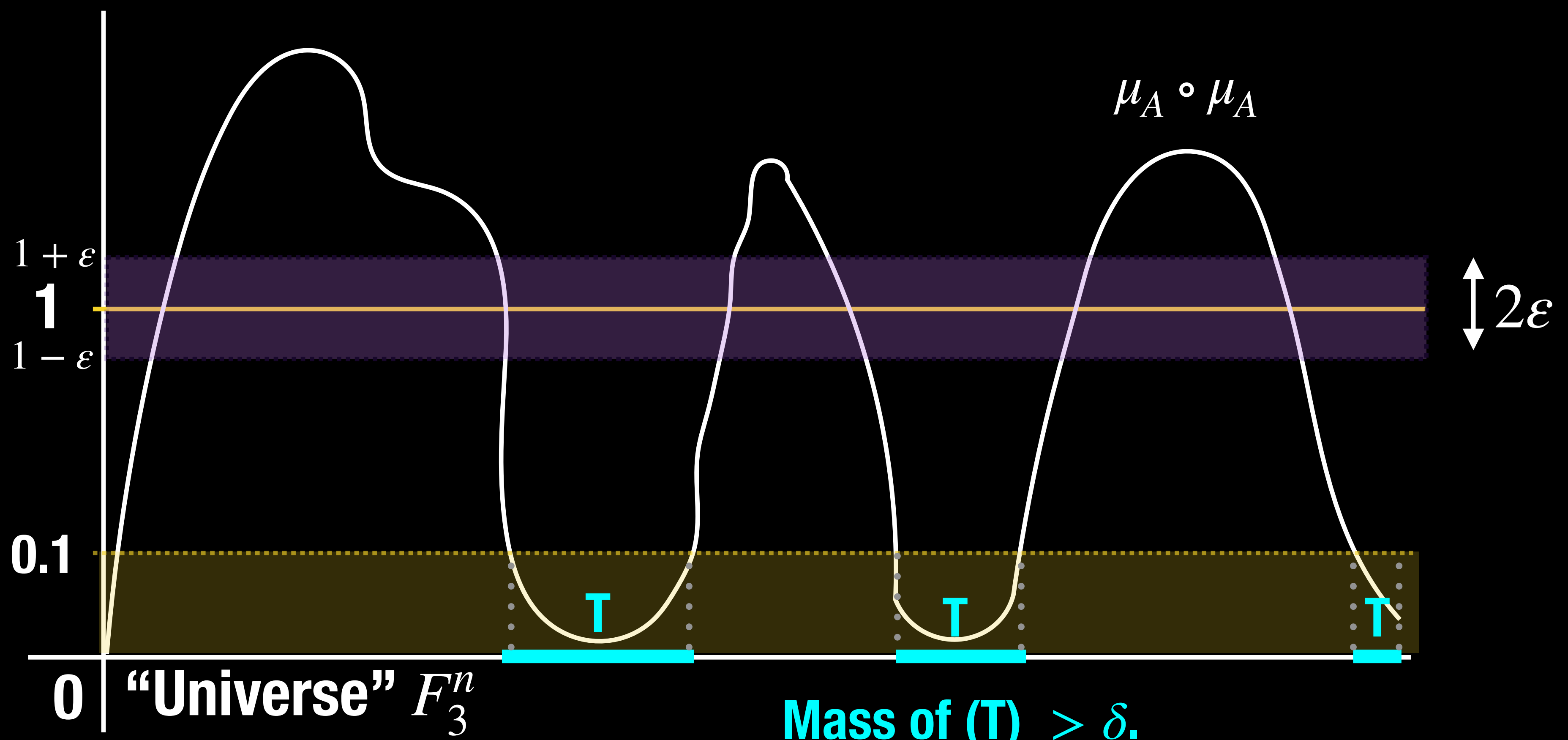

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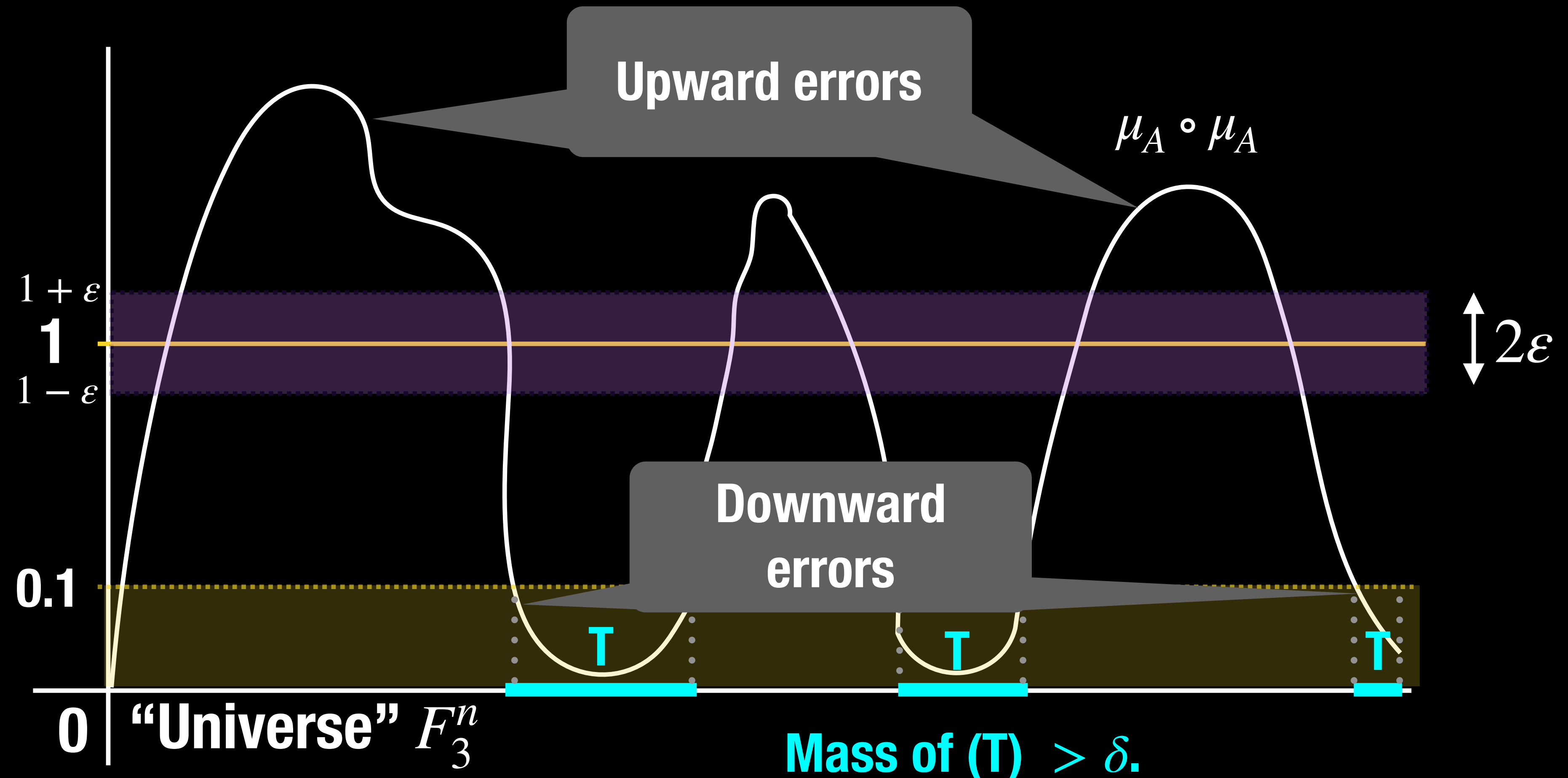

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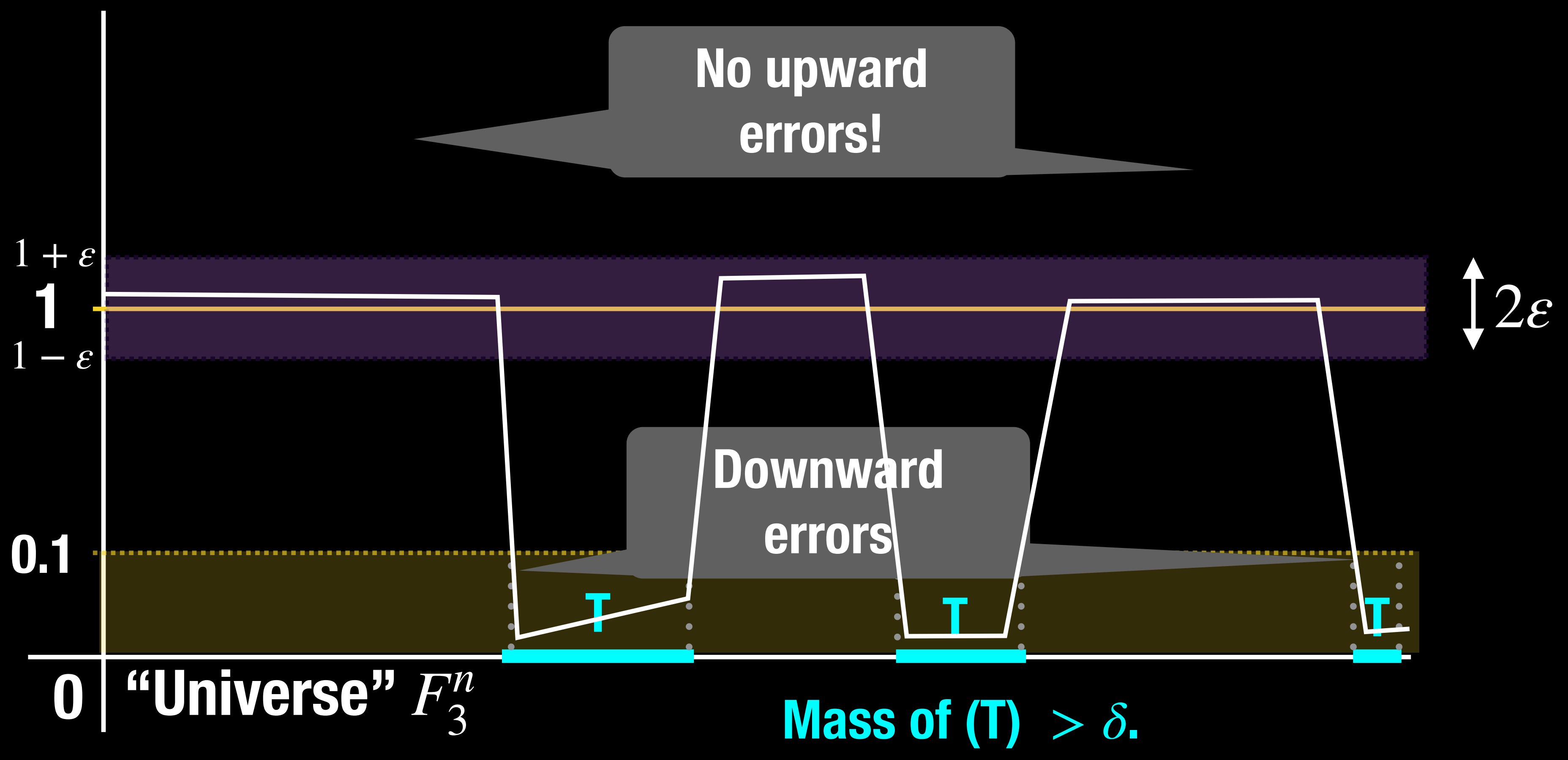
Spectral Positivity

General densities can have only downward errors.



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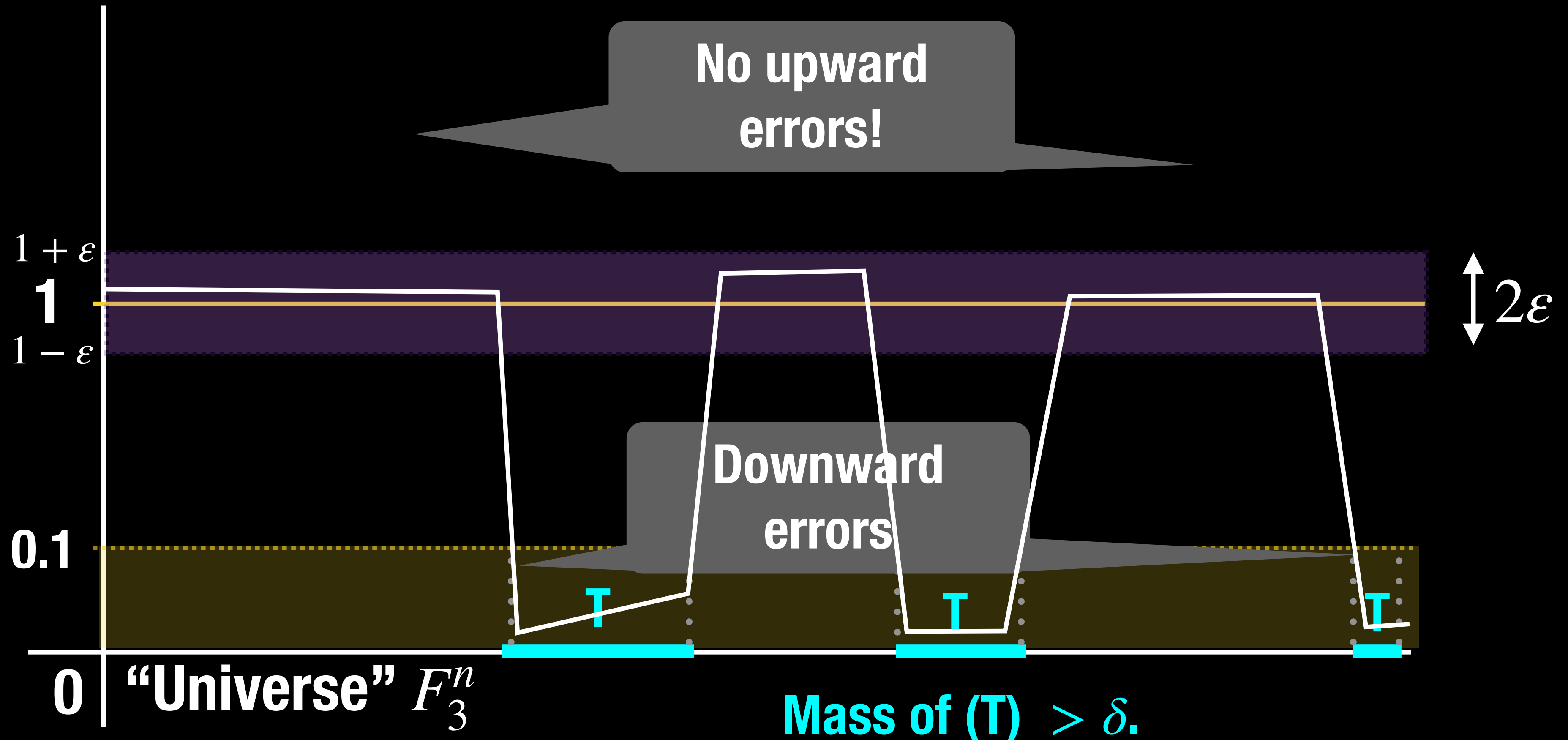
[KM23]: A spectrally positive density, cannot have only downward errors.



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Spectrally positive: Fourier coefficients non-negative

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Spectral Positivity

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Spectrally positive: Fourier coefficients non-negative

Fact: Fourier coefficients of $\mu_A \circ \mu_A$ are given by

$$\widehat{\mu_A \circ \mu_A}(w) = |\widehat{\mu_A}(w)|^2 .$$

(Convolution is product in Fourier domain)

$A \subseteq \{0,1,2\}^n$

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Spectral Positivity: Summary

Why useful? We want density increment
i.e., we want V for which we lower bound $|A \cap V|$.

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Spectral Positivity: Summary

SP Lemma: $\mu_A \circ \mu_A$ **not lower-concentrated.**
Then, it must also take large values.

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Density increment onto subspaces

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Sifting: Finding additive structure

“Final" goal: $\mu_A \circ \mu_A$ **takes large values often**
$\implies$ **Density increment.**

Want: A large affine space V with

$$|A \cap V| > (1.01)\,\delta\,|V|\,.$$

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Want: A large set B with,

$$|A \cap (B+B)| > 1.05 \cdot \delta \cdot |B+B|?$$

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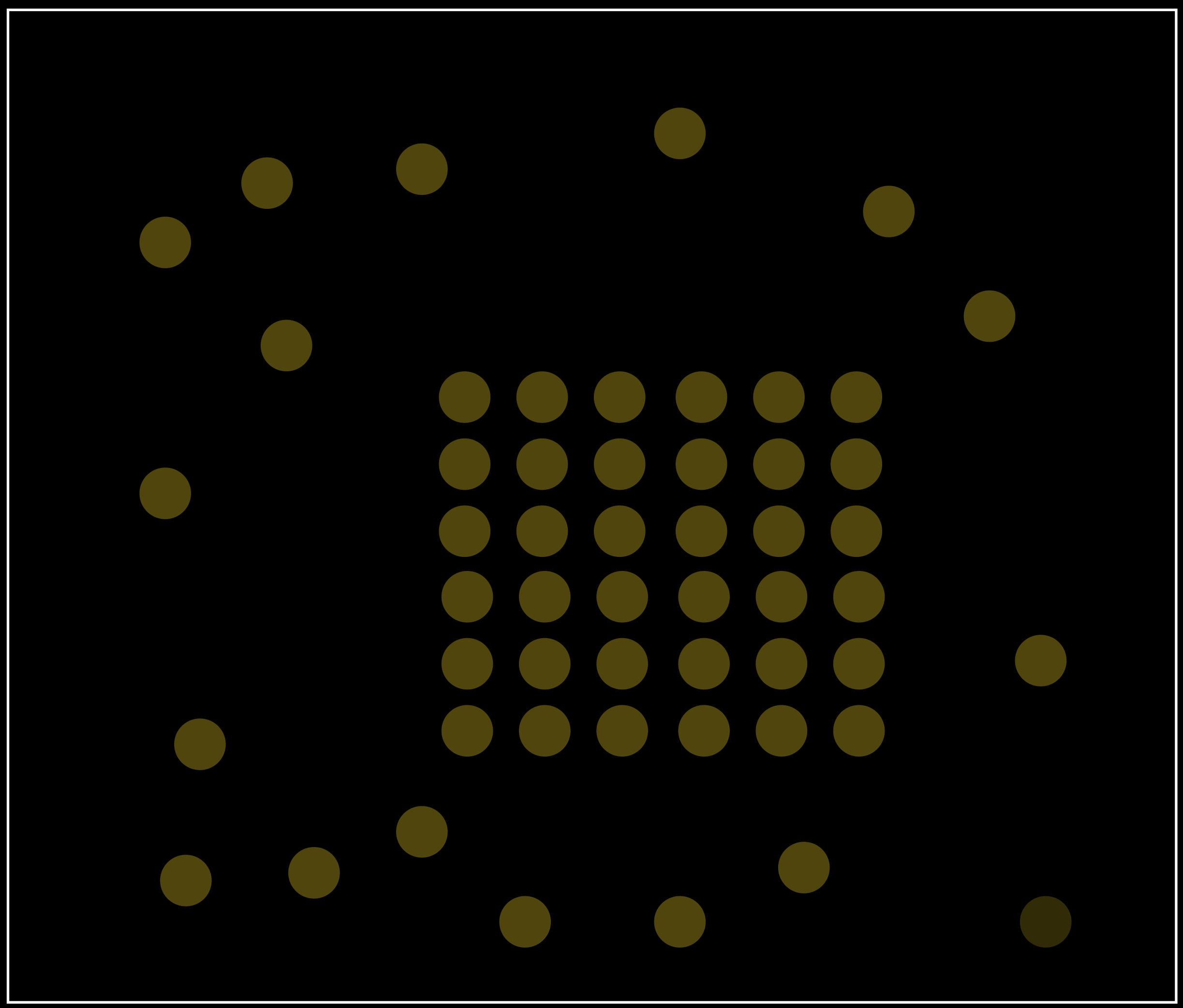
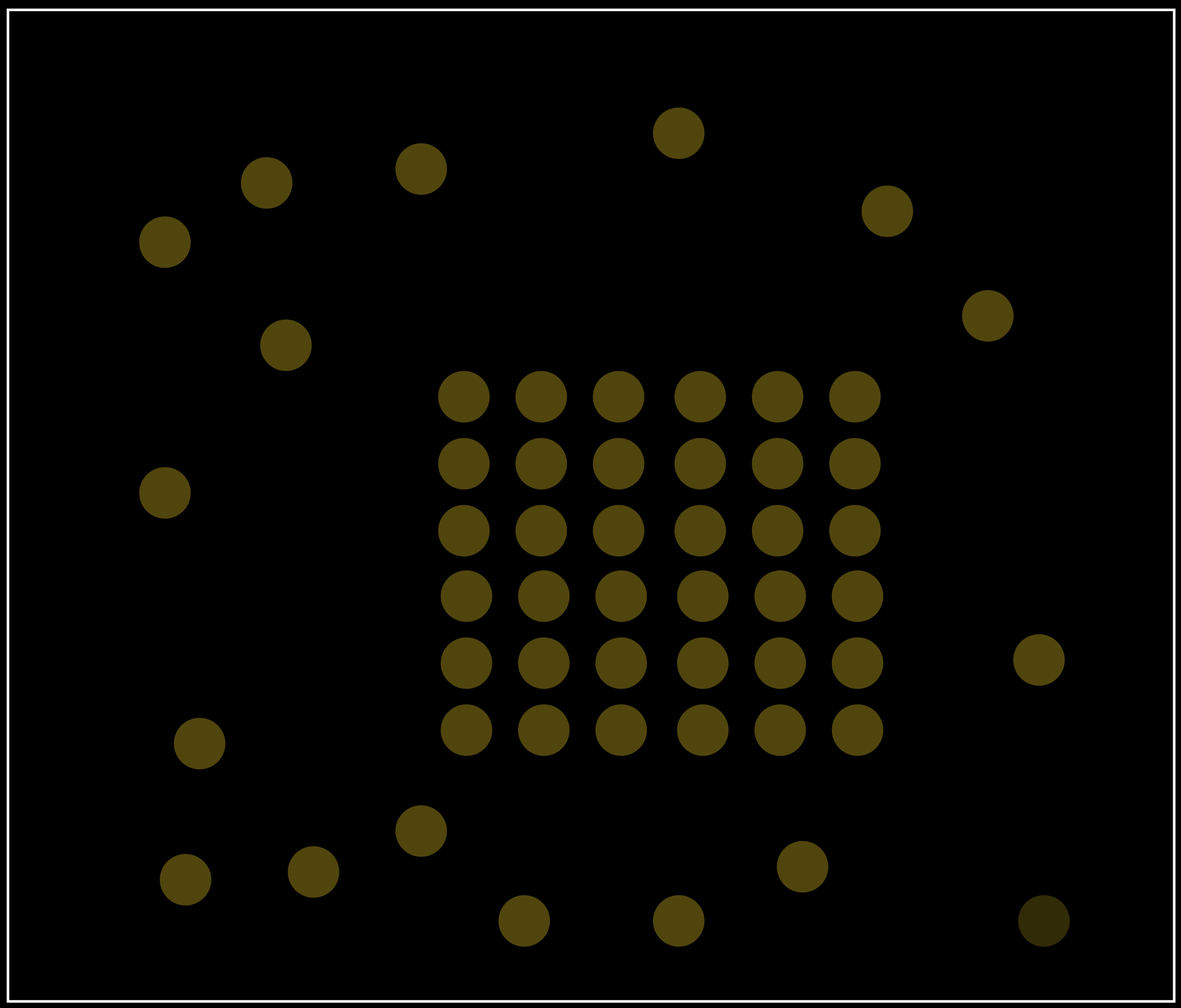


Large set $S \subseteq \mathbb{F}_3^n$.
What "operation" will "sift-out" the portion with structure?

Sifting



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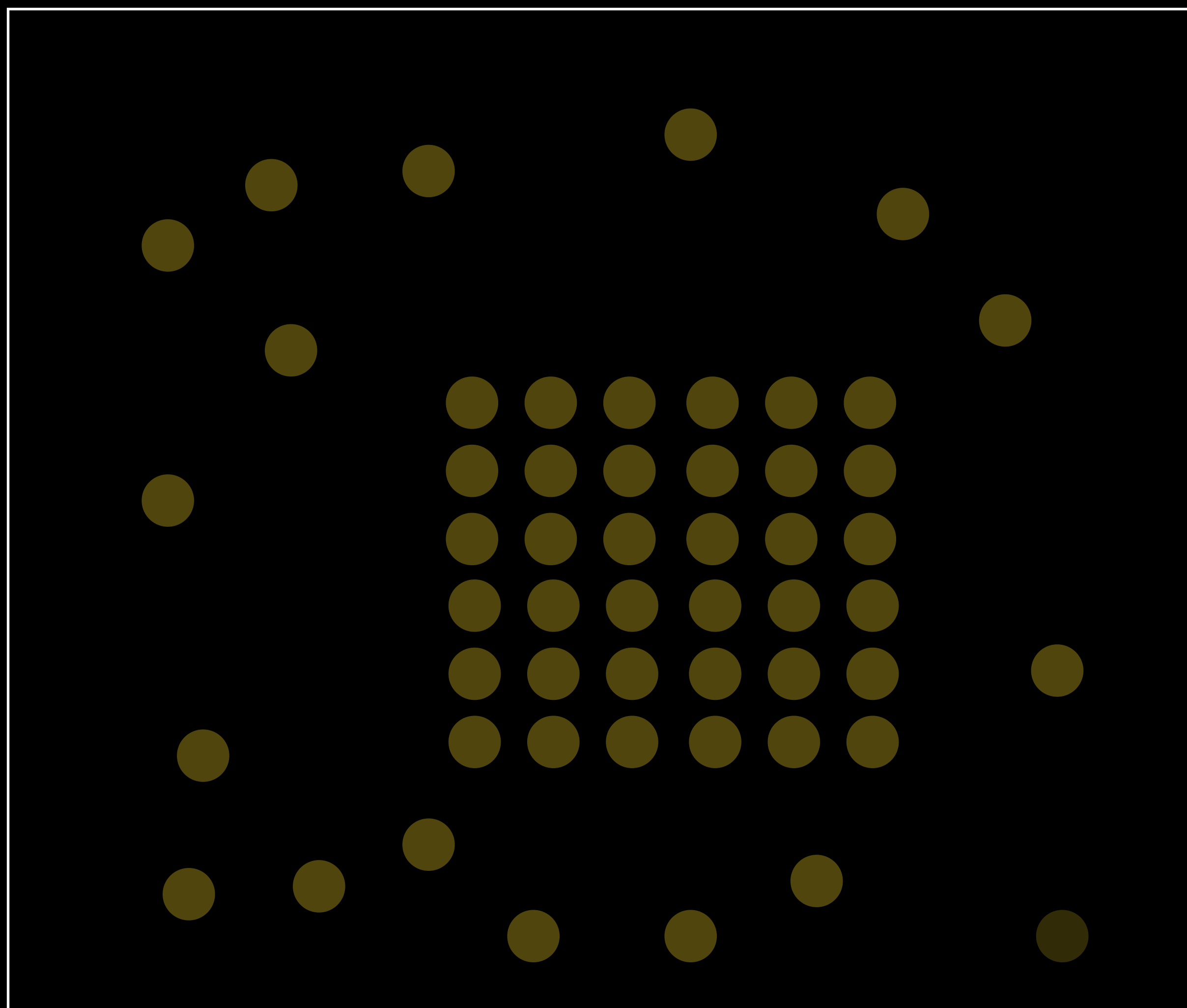

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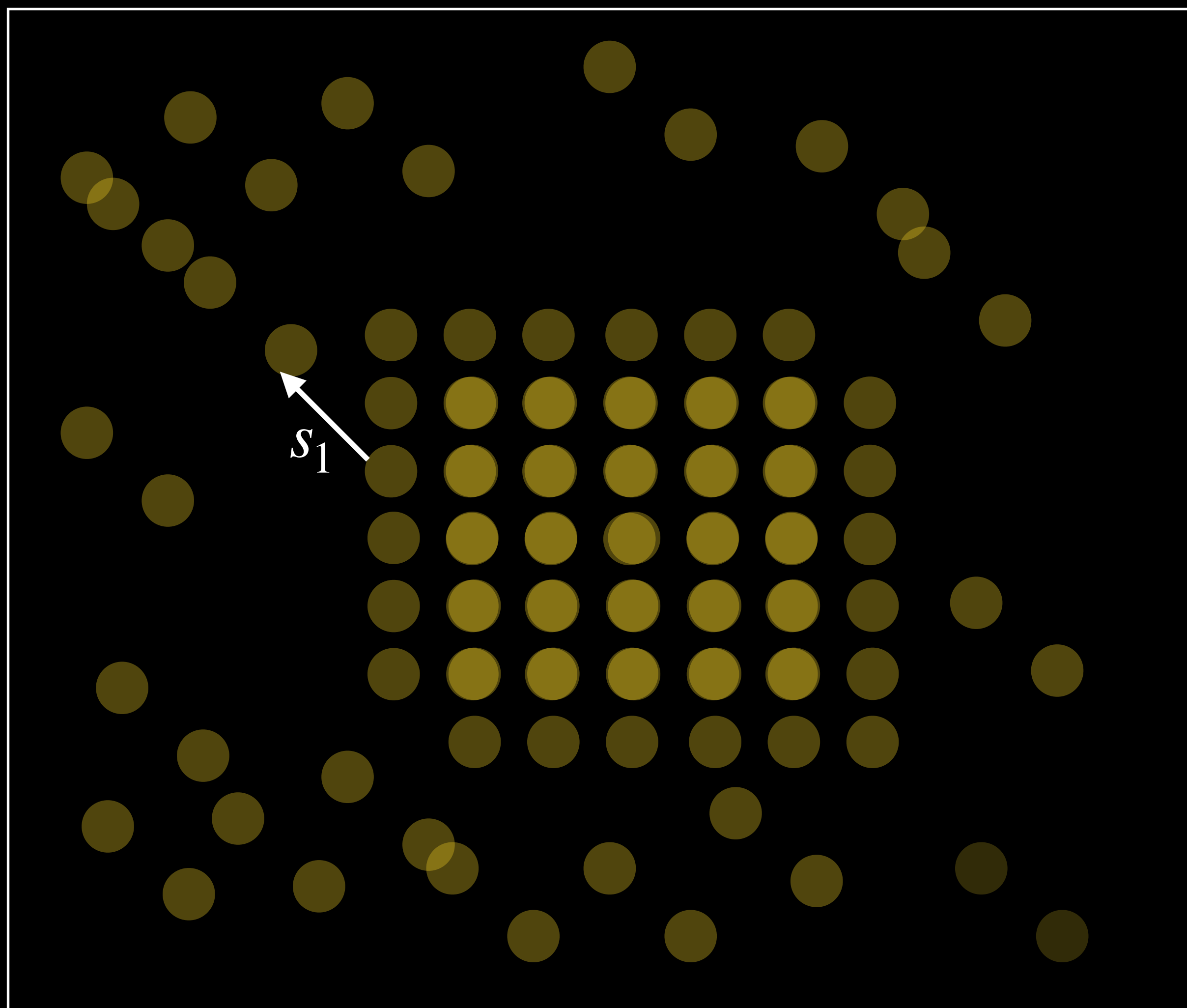
Random $s_1, s_2, \ldots, s_\ell$

Sifting



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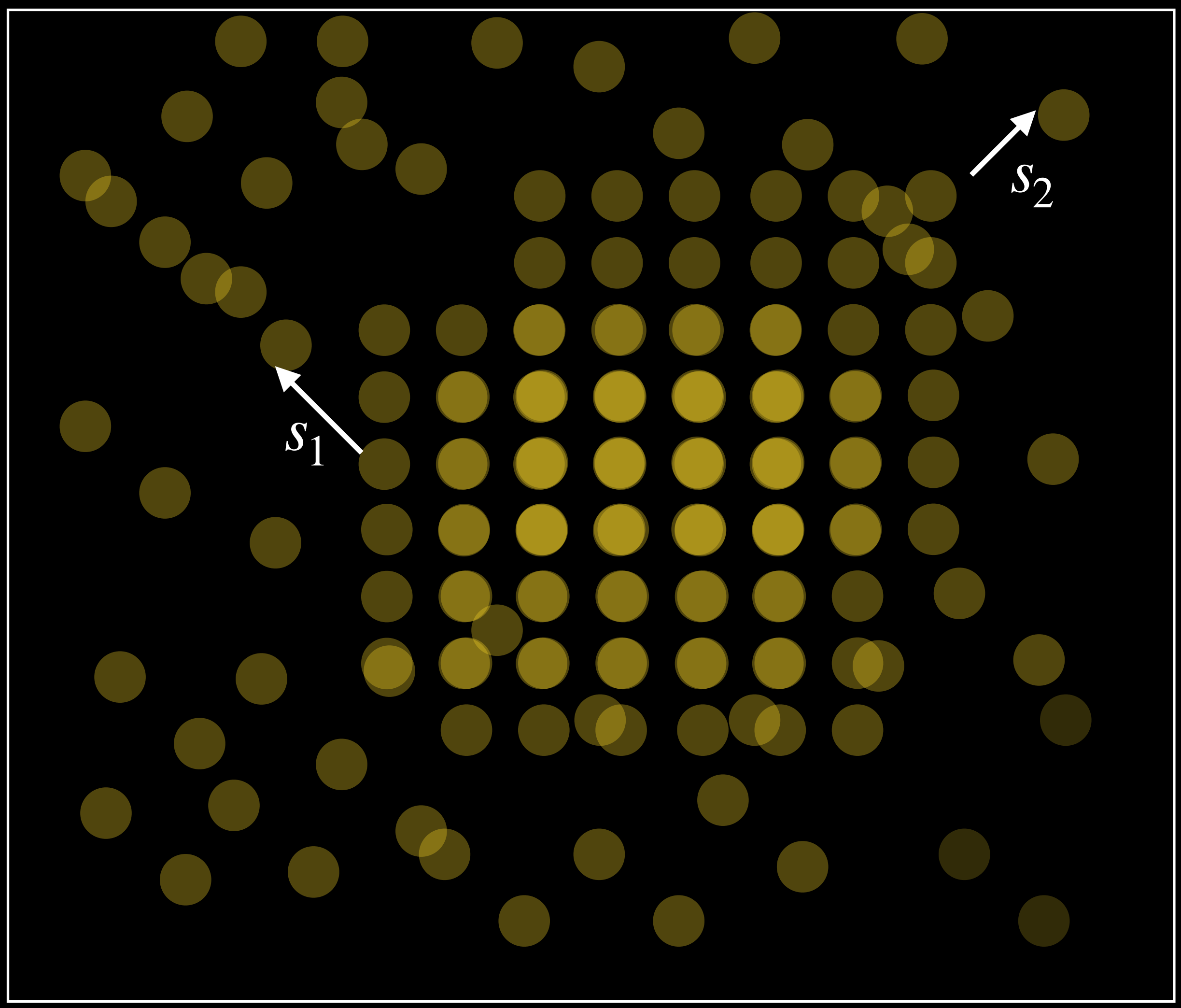

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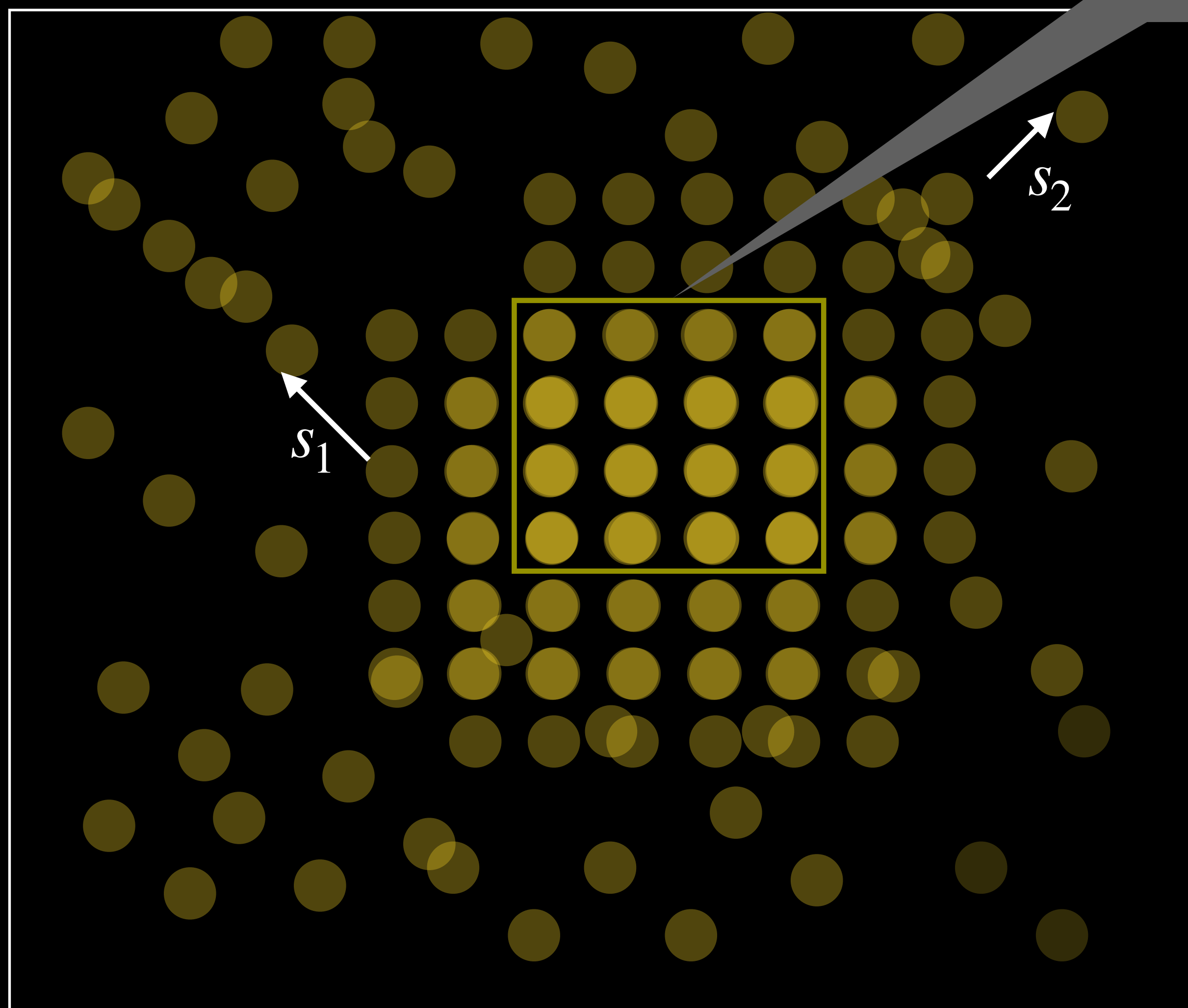
Bigger fraction of inner-square will survive than from isolated points.



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Sifting: Finding Correlated Sumset

“Final" goal: $\mu_A \circ \mu_A$ takes large values often
$\Longrightarrow$ Density increment.

Sifting Lemma: There exists a large B as below such that we have density increment on B+B.

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μ_A **density of A**

Sifting: Finding Correlated Sumset

Sifting Lemma: There exists a large B as below such that we have density increment on B+B.

Key identity: Can exactly count what sifting does!

$$N^2 \cdot \|\mu_A \circ \mu_A\|_k^k = E[\,|(A+s_1) \cap \cdots \cap (A+s_k)|^2\,].$$

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Look at $B = S \cap (S+s_1) \cap (S+s_2) \cap \cdots (S+s_\ell)$

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Almost Periodicity: Sumsets to Subspaces

"Final" goal: $\mu_A \circ \mu_A$ **takes large values often**
$\implies$ **Density increment.**

Sanders' Theorem: If $\exists B$ **of density** $2^{-O(\ell)}$**,**
$B + B \approx S$**. Then,** S **has density increment onto an affine space V with co-dim(V)** $= O(\ell^4)$ **.**

Sifting Summary

Sifting: $\|\mu_A \circ \mu_A\|_k > 1 + \varepsilon \Rightarrow$
There is an affine space V of co-dimension
$O(k^4 \log(1/\delta)^4)$, $|A \cap V| > (1.01) \cdot \delta |V|$.

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Proof Summary

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SP Lemma: If $\mu_A \circ \mu_A$ **not lower-concentrated,**
then $\|\mu_A \circ \mu_A\|_k > 1 + .01$, $k = O(\log(1/\delta))$.

S vs R Lemma: $\mu_A \circ \mu_A$ **lower-concentrated, then**
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Summary

We have a subset A of $\{0,1,2\}^n$, $|A| = \delta 3^n$.

Are there unequal $a, b, c \in A$, $a + b = 2c \mod 3$?

An analytic proof that $\delta \sim 2^{-\Omega((\log N)^{0.11})}$ suffices.

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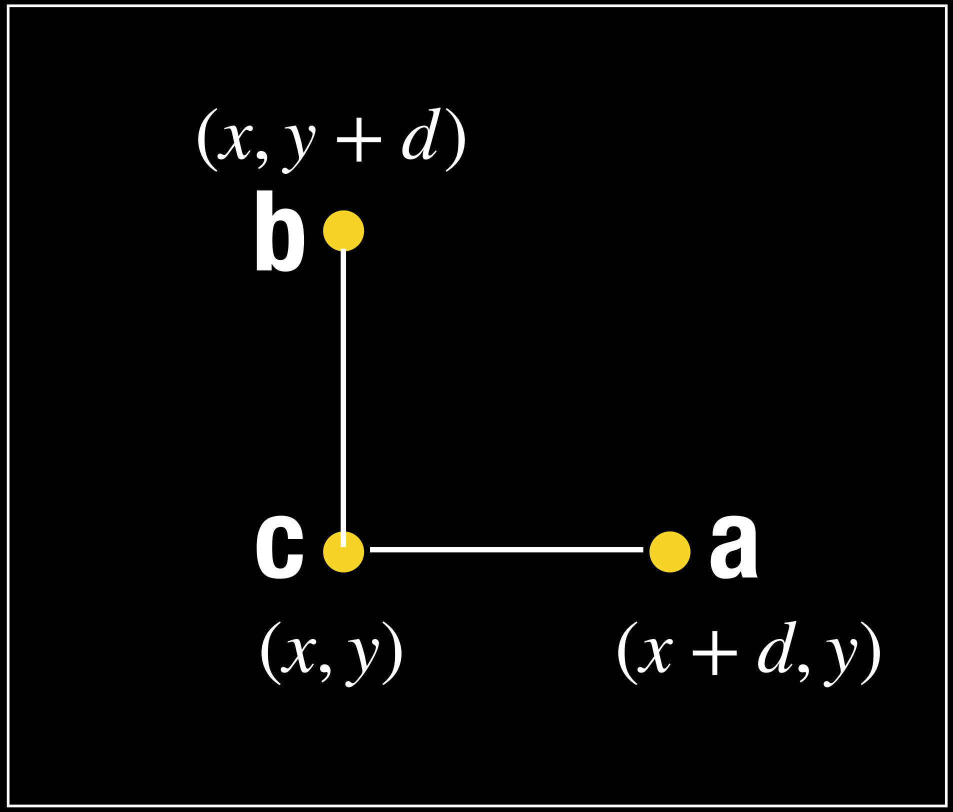
Going from finite fields to integers is quite technical but uses similar ideas.

Main Thm: Any set of at least δN integers from [N] for $\delta > 2^{-\Omega((\log N)^{.08})}$ has many 3APs.

What's next?

1. Four-term arithmetic progressions?

2. "Corners problem": $A \subseteq [N]^2, |A| = \delta N^2$. How small can δ be so that A always have a "corner"?



Behrend: Need $\delta > 2^{-O(\sqrt{\log N})}$

Corner = $\{(x,y), (x+d,y), (x,y+d)\}$

THANK YOU

Going beyond?

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Wish 2: Far from uniform $\Rightarrow$ affine space V with density increment , co-dim(V) small.

$A \subseteq \mathbb{F}_3^n$
$|A| = \delta N, N = 3^n$.

Going beyond?

Wish 1: "Close to" uniform
$\Rightarrow$ Many 3APs

Wish 2: Far from uniform $\Rightarrow$ affine space V
with density increment , co-dim(V) small.

Step 1. "Close" to uniform $\Rightarrow$ Many 3APs?

Step 2. Far from uniform $\Rightarrow$ affine V with
(1.01) **density increment,**
co-dim$(V) \leq poly(\log(1/\delta))$?

Roth-Meshulam argument:
Need co-dim$(V) \approx 1/\delta$.

$A \subseteq \mathbb{F}_3^n$
$|A| = \delta N, N = 3^n$.

Sifting: Finding additive structure

"Final" goal: $\|\mu_A \circ \mu_A\|_k > 1 + \varepsilon \Rightarrow$
(Strong) Density increment

Want: A large affine space V with
$$|A \cap V| > (1.01)\,\delta\,|V|\,.$$

$A \subseteq \mathbb{F}_3^n$
$|A| = \delta N, N = 3^n\,.$
μ_A **density of A**

Sifting: Finding additive structure

“Final" goal: $\|\mu_A \circ \mu_A\|_k > 1 + \varepsilon \Rightarrow$
(Strong) Density increment

Want: A large affine space V with

$$|A \cap V| > (1.01)\,\delta\,|V|.$$

“Forget” full affine space V, how about partial progress?

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Sifting: Finding additive structure

“Final" goal: $\|\mu_A \circ \mu_A\|_k > 1 + \varepsilon \Rightarrow$ **(Strong) Density increment**

Want: A large affine space V with
$$|A \cap V| > (1.01)\,\delta\,|V|\,.$$

How about: Find increment onto a sumset?
Large B, $|A \cap (B+B)| > 1.05 \cdot \delta \cdot |B+B|\,?$

$A \subseteq \mathbb{F}_3^n$
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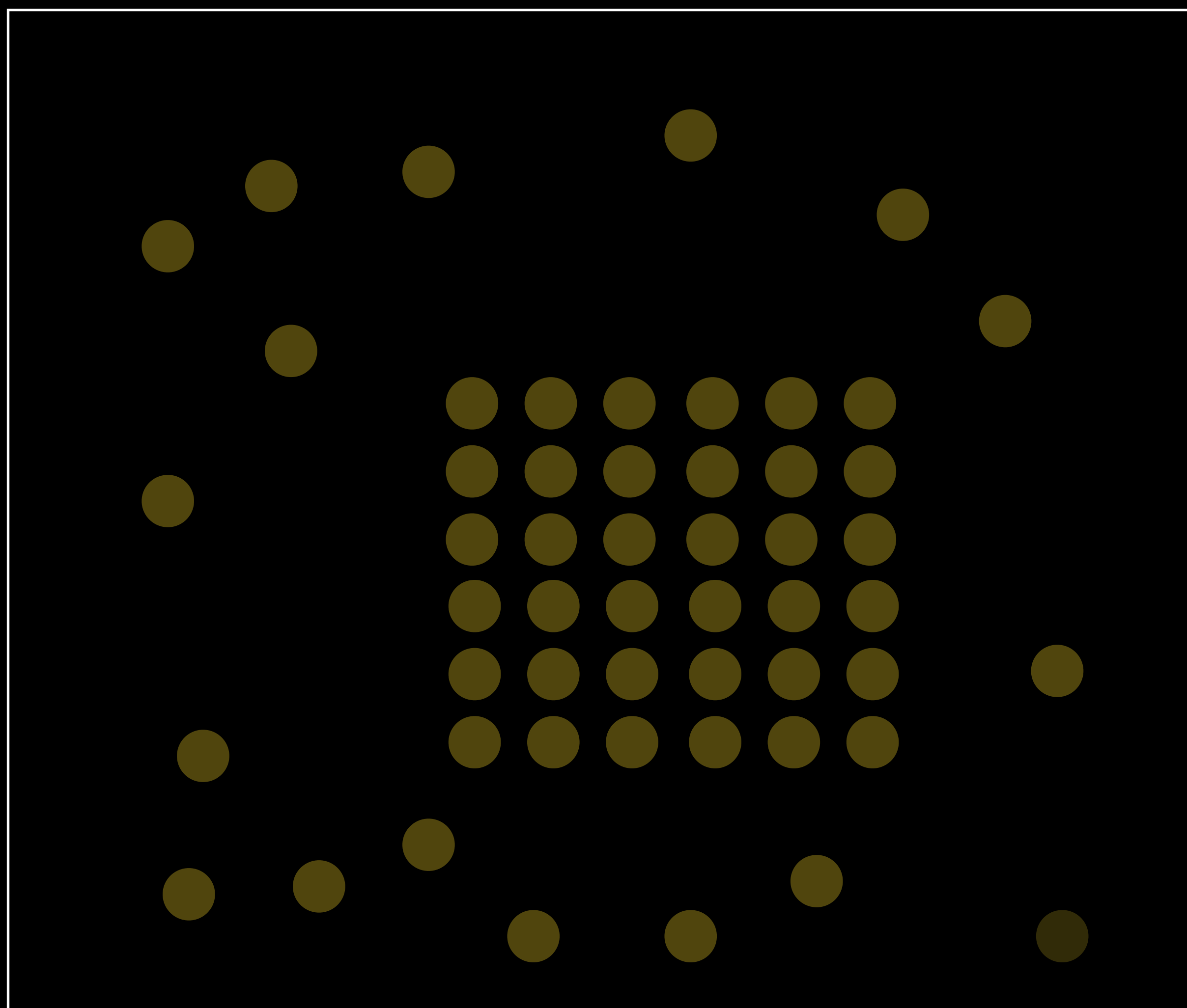
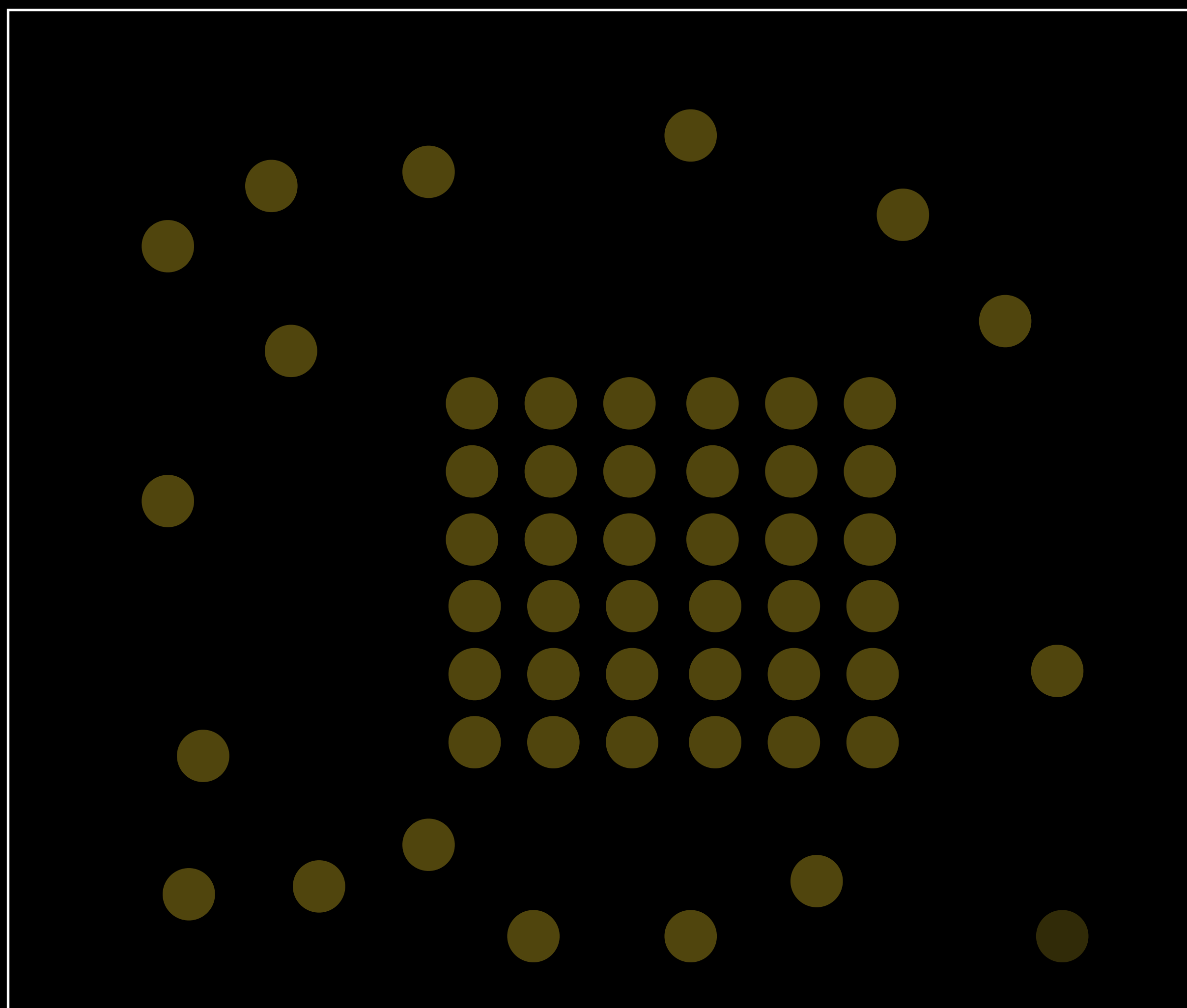
Sifting



Large set $S \subseteq \mathbb{F}_3^n$.

What "operation" will "sift-out" the portion with structure?

Sifting



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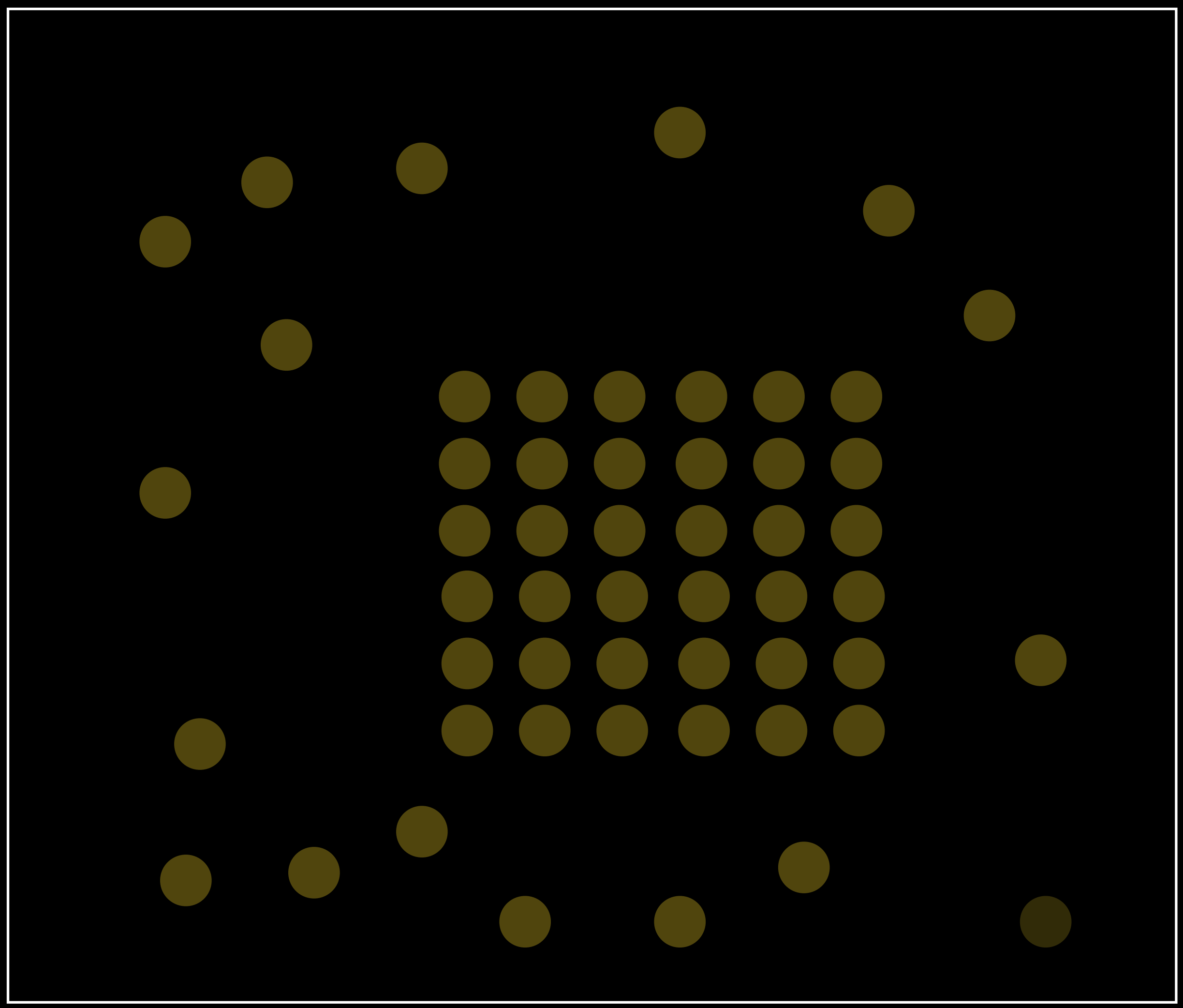
Idea: Shift and intersect!

Sifting



Large set $S \subseteq \mathbb{F}_3^n$.

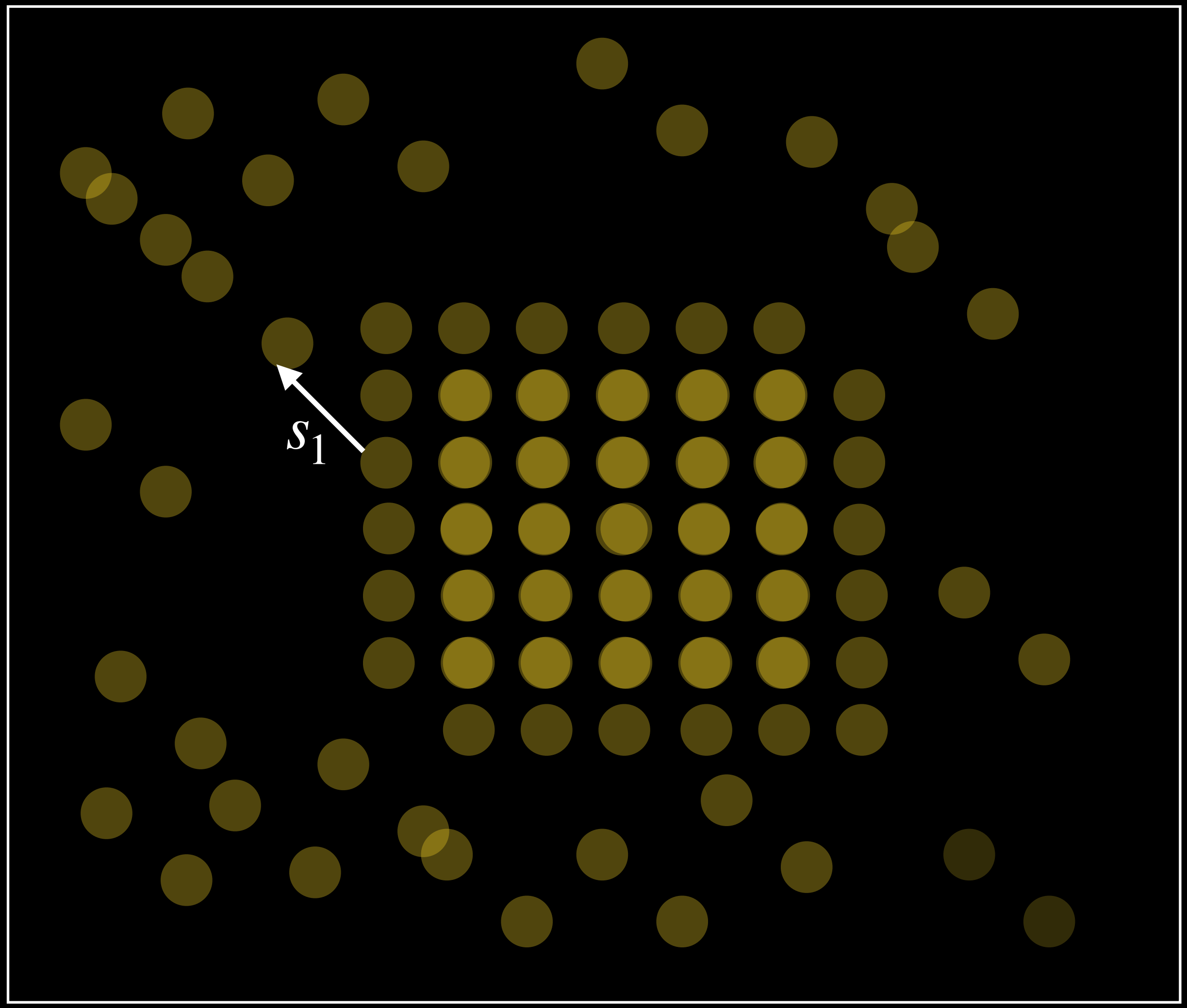
What "operation" will "sift-out" the portion with structure?

Idea: Shift and intersect!

Look at $B = S \cap (S + s_1) \cap (S + s_2) \cap \cdots (S + s_\ell)$

Random $s_1, s_2, \ldots, s_\ell$

Sifting



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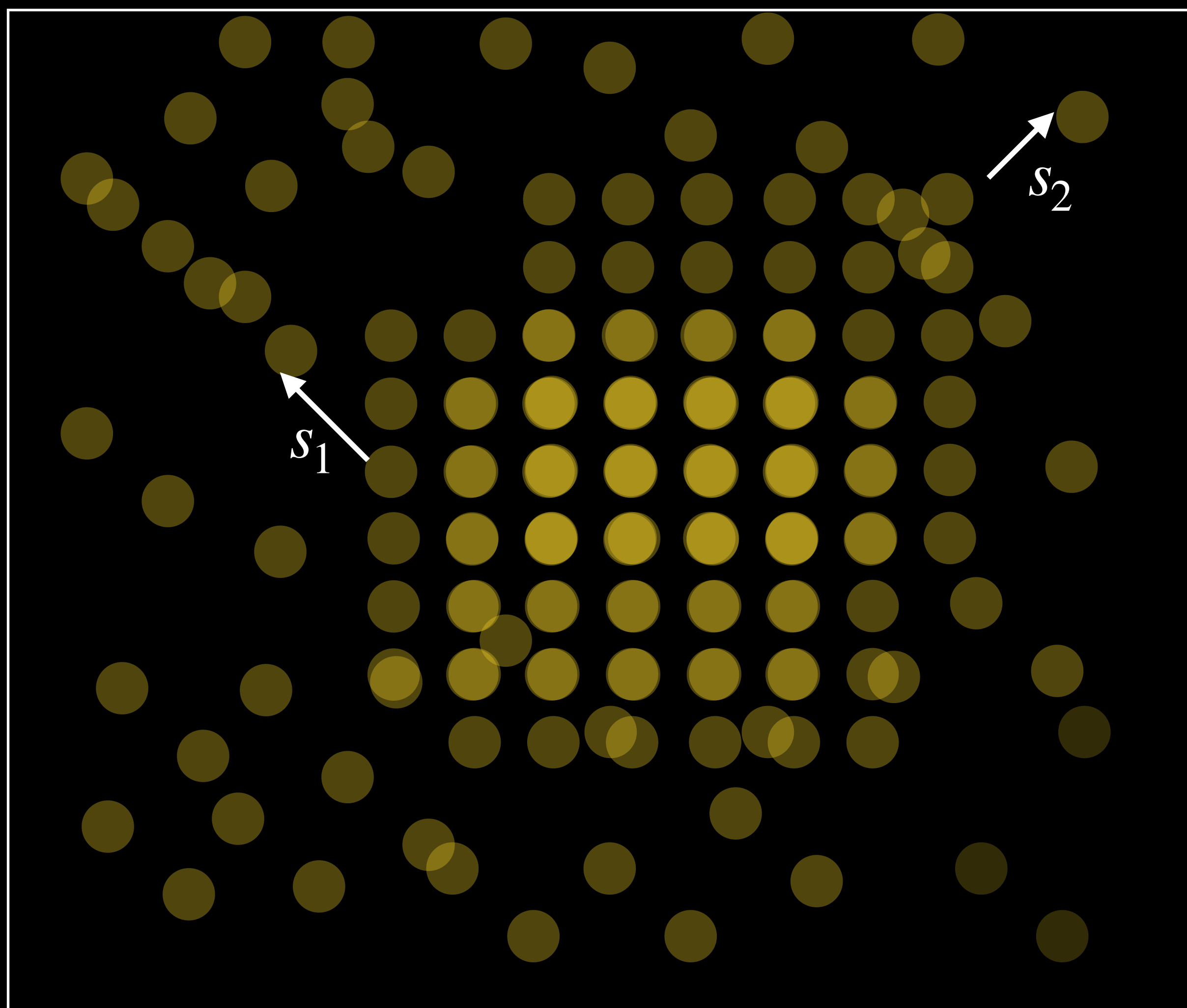

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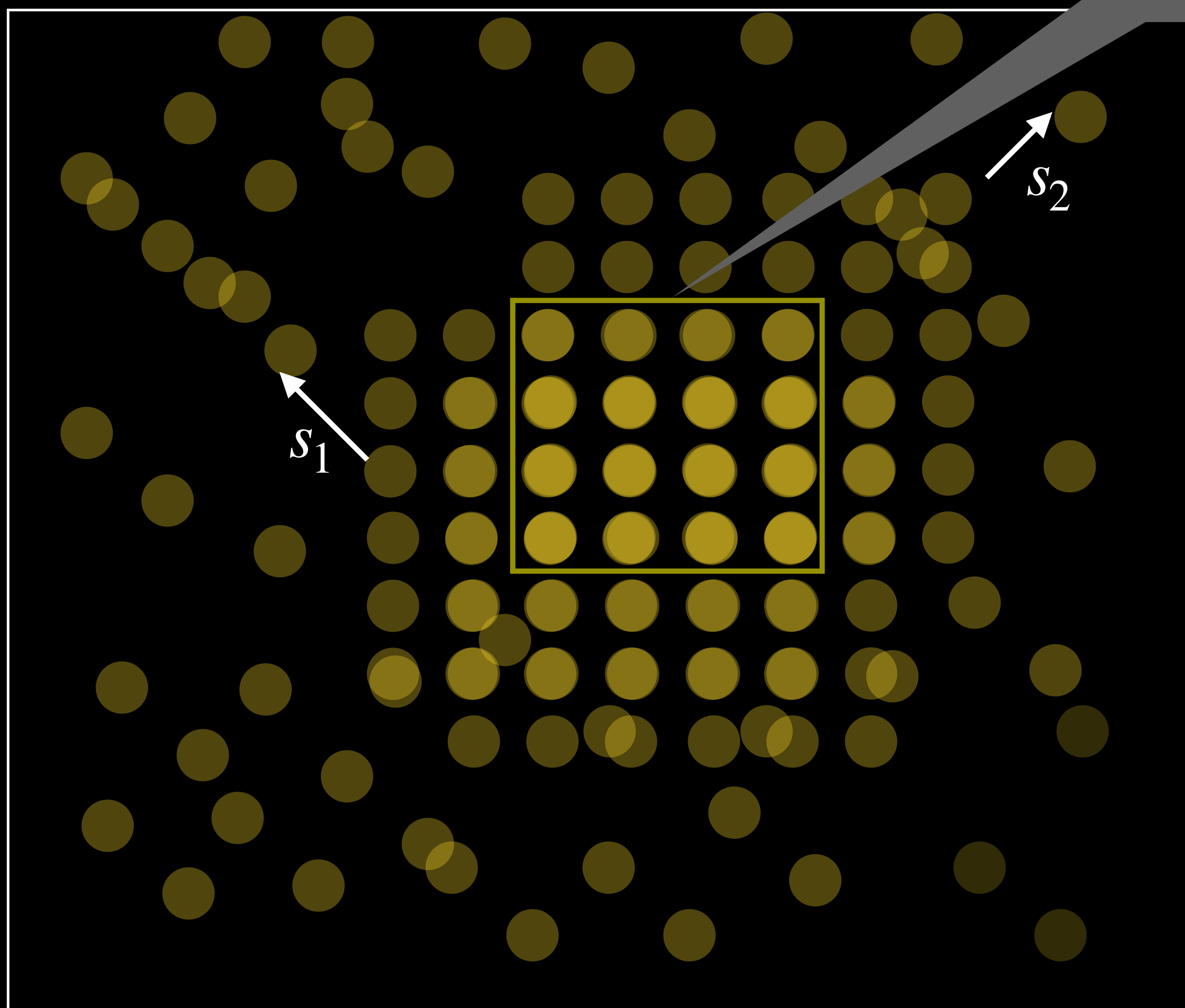

Bigger fraction of inner-square will survive than from isolated points.

Large set $S \subseteq \mathbb{F}_3^n$.
What "operation" will "sift-out" the portion with structure?

Idea: Shift and intersect!

Look at $B = S \cap (S + s_1) \cap (S + s_2) \cap \cdots (S + s_\ell)$

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Sifting: Finding Correlated Sumset

"Final" goal: $\|\mu_A \circ \mu_A\|_k > 1 + \varepsilon \Rightarrow$
(Strong) Density increment

Sifting Lemma: $S = \{z : \mu_A \circ \mu_A(z) > 1 + \varepsilon/4\}$.

$\exists B \subset \mathbb{F}_3^n$, $|B| = \delta^{O(k)}N$, $B + B \approx S$.
$X \sim \mu_B, Y \sim \mu_B$, $Pr[X + Y \in S] \geq 1 - O(\varepsilon)$.

Idea: Shift and intersect!

Look at $B = S \cap (S + s_1) \cap (S + s_2) \cap \cdots (S + s_\ell)$

Random $s_1, s_2, \ldots, s_\ell$

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Key identity

$$N^2 \cdot \|\mu_A \circ \mu_A\|_k^k = E[\,|(A + s_1) \cap \cdots \cap (A + s_k)|^2\,] .$$

Idea: Shift and intersect!

Look at $B = S \cap (S + s_1) \cap (S + s_2) \cap \cdots (S + s_\ell)$

Random $s_1, s_2, \ldots, s_\ell$

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Sanders' Invariance: Sumsets to Subspaces

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$X \sim \mu_B, Y \sim \mu_B, Pr[X + Y \in S] \geq 1 - O(\varepsilon)$.

Sanders' invariance: If $\exists B$ **of density** $2^{-O(\ell)}$**,**
$B + B \approx S$**. Then,** S **has density increment onto an affine space V with co-dim(V)** $= O(\ell^4)$.

Sifting Summary

Sifting: $\|\mu_A \circ \mu_A\|_k > 1 + \varepsilon \Rightarrow$
There is an affine space V **of co-dimension**
$O(k^4 \log(1/\delta)^4)$**,** $|A \cap V| > (1.01) \cdot \delta |V|$.

$A \subseteq \mathbb{F}_3^n$
$|A| = \delta N, N = 3^n$.
μ_A **density of A**

Proof Summary

Sifting: $\|\mu_A \circ \mu_A\|_k > 1 + \varepsilon \Rightarrow$
There is density increment with good parameters.

SP Lemma: If $\mu_A \circ \mu_A$ **not lower-concentrated,**
then $\|\mu_A \circ \mu_A\|_k > 1 + .01, k = O(\log(1/\delta))$**.**

S vs R Lemma: $\mu_A \circ \mu_A$ **lower-concentrated, then**
we have many 3APs.

$A \subseteq \mathbb{F}_3^n$
$|A| = \delta N, N = 3^n.$
μ_A **density of A**