Matchings with Fairness Constraints[1]

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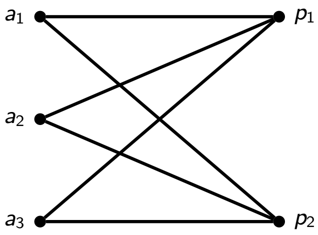
Recent Trends in Algorithms, 2023
NISER Bhubaneswar

[1] Based on joint results with Anand Louis, Meghana Nasre, Atasi Panda, Nada Pulath, Govind Sankar

Problem setup

Input:

- A set of items
- A set of platforms



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- A set of items
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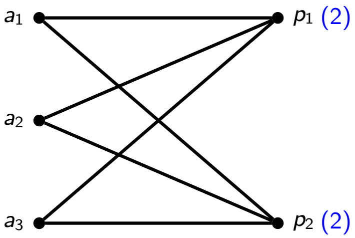
- Platforms have quotas



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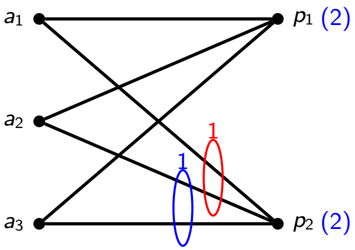

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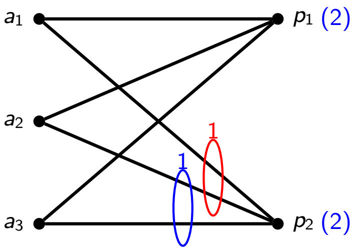

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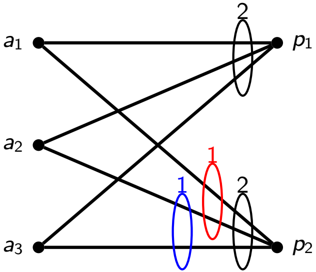
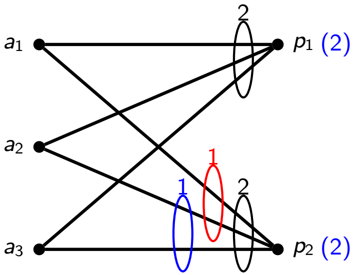


Classes are subsets on the neighborhood.
Classes denote fairness constraints.
The neighborhood is a trivial class!



Goal: Match maximum number of items to platforms

Why classes?

Some natural constraints that can be modelled:

Selection of committees

- Committee - Needs to have experts from all areas

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Selection of committees

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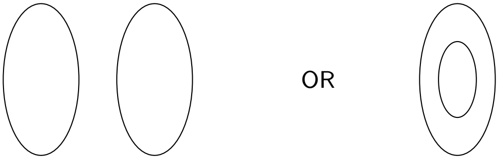
Formation of teams for projects

- Project - wasteful to have many employees with the same skills.

A Special Case: Laminar classes

Huang (2010); 2-sided pref.

Laminar classification $\iff$ any pair of classes has no nontrivial intersection



- Example: Countries , States , Districts , Cities
- Special case: Partition i.e. disjoint classes

Reduction to Max-Flow Problem

Maximum matchings under laminar classes

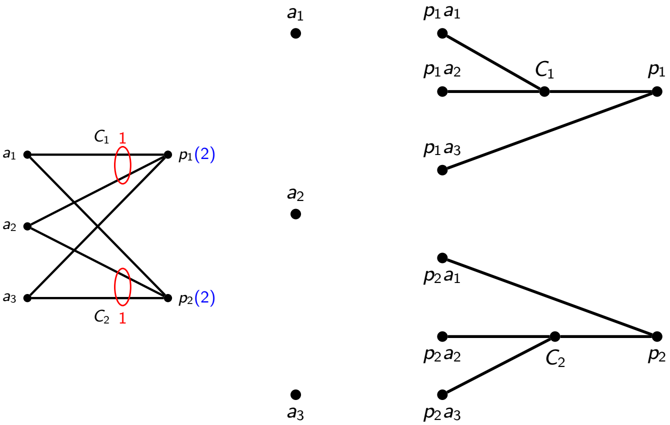
Reduction to Max-Flow Problem

Maximum matchings under laminar classes

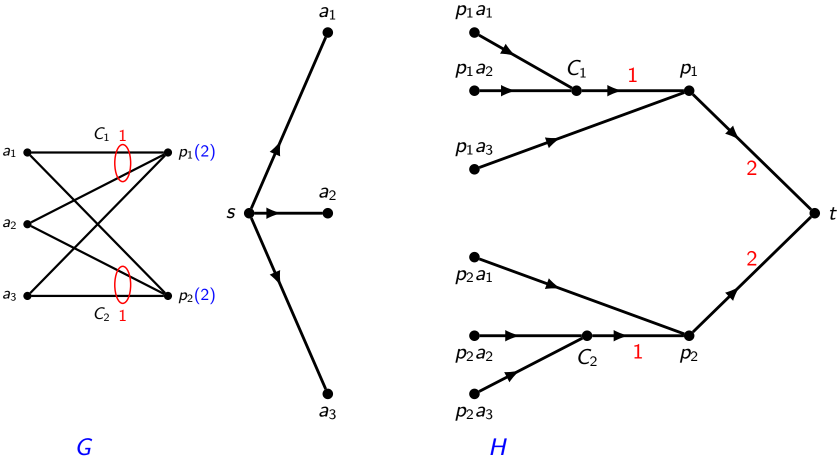
$\Downarrow$

Maximum flow in a flow network

Classification tree - property of laminar classification

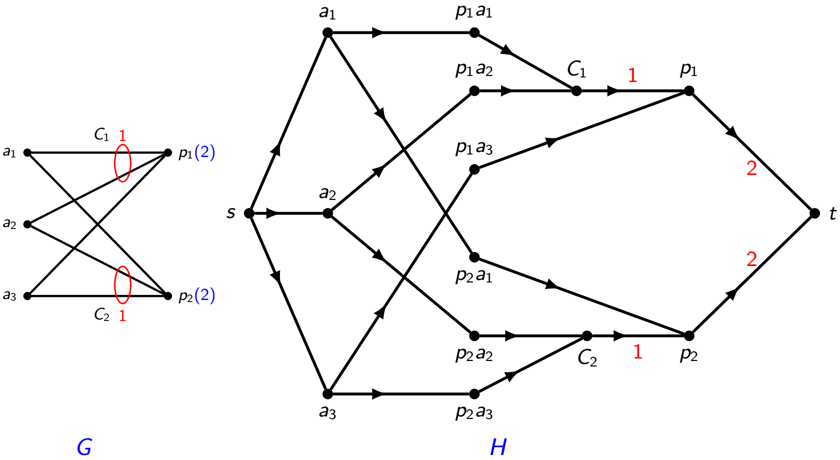


Feasible matchings to feasible flows



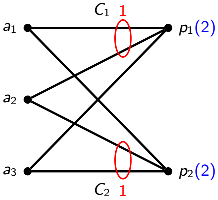
Add source s and sink t

Feasible matchings to feasible flows

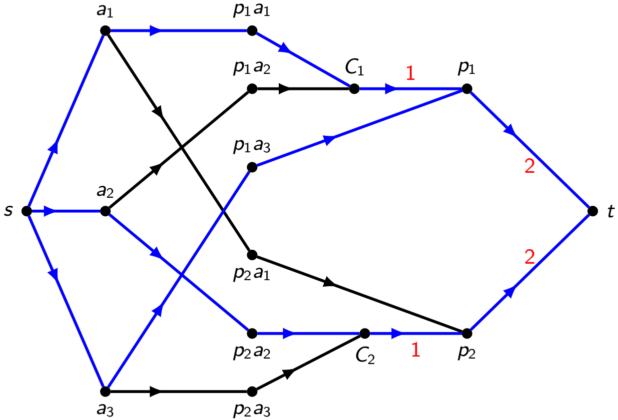


Add (a, pa) edges

Feasible matchings to feasible flows



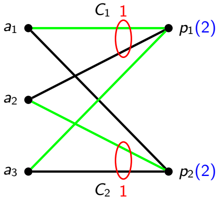
G



H

Compute a max-flow in H

Feasible matchings to feasible flows



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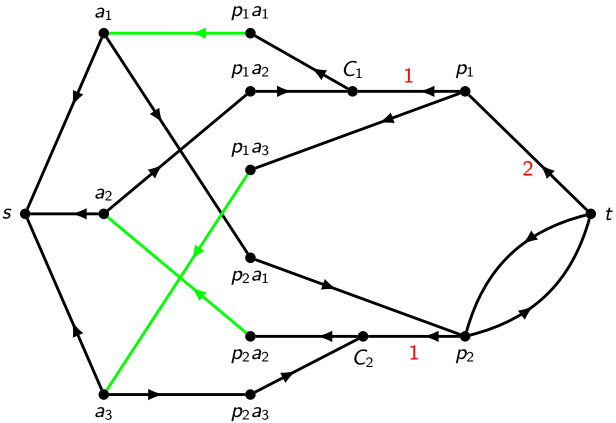


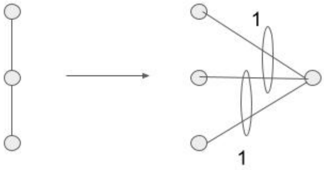
H

Maximum matching in $G \iff$ Maximum flow in H

Hardness for non-laminar classes

Reduction from independent set problem

- Vertices → items
- Only one platform
- Complete bipartite graph
- Edges → classes with quota 1



Hardness continued...

Independent set $\equiv$ Matching respecting class quotas

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No $n^{\epsilon-1}$-approximation for any $\epsilon > 0$ unless P=NP [Zuckerman]

$O(1)$ classes: $\frac{1}{2}$-approximation [Sankar, Louis, Nasre, N. IJCAI'21]

$\Delta = O(1)$ classes, one platform $\Rightarrow$ Exact algorithm by solving ILP

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ILP:

$$\text{Maximize} \quad \sum_{i=1}^{2^{\Delta}} x_i$$

$$\text{Subject to}$$

$$\sum_{C \in \text{Type } i} x_i \leq q(C) \quad \text{for each class } C$$

$$0 \leq x_i \leq \text{number of Type } i \text{ items}$$

From one platform to multiple platforms

Theorem

α-approximation for one platform $\Rightarrow \frac{\alpha}{1+\alpha}$-approximation for multiple platforms.

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$\frac{1}{2}$-approximation for multiple platforms, $O(1)$ classes

Proof of the theorem

Algorithm: Find α-approximation for each platform, from remaining items.

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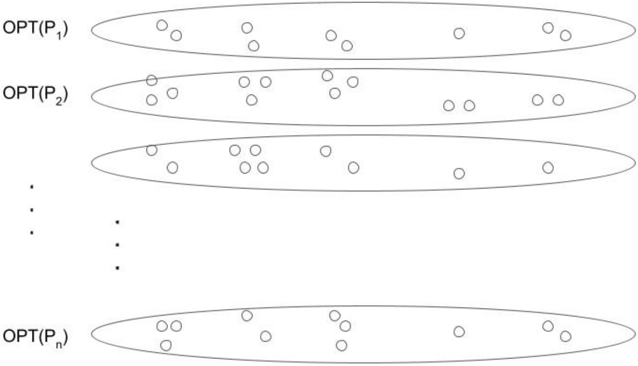

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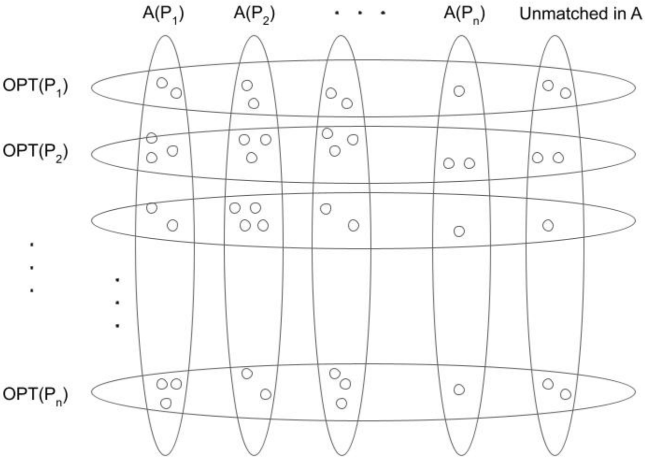

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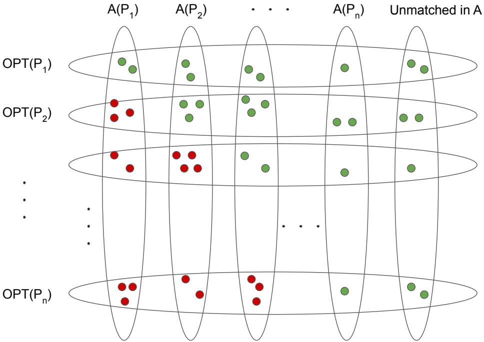

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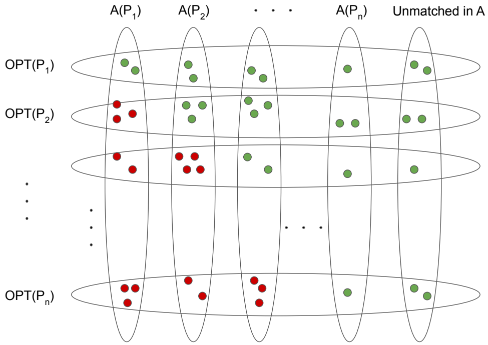

$|OPT| = |Red| + |Green| \qquad |A| \geq \alpha|Green| \qquad |A| \geq |Red|$

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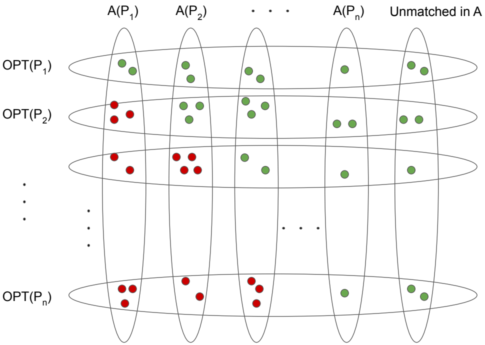

$$|OPT| = |Red| + |Green| \qquad |A| \geq \alpha|Green| \qquad |A| \geq |Red|$$

$$(1+\alpha)|A| \geq \alpha|OPT|$$

Another simple case

When each item is in $\leq \Delta$ classes: $\frac{1}{\Delta+1}$-approximation

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Simple maximal matching like argument.

Connection with hypergraph independent sets

Hypergraph: (Hyper)edges are sets of k vertices ($k = 2$ for graphs)
Max-degree $= \Delta$

How to define independent set for hypergraphs?

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How to define independent set for hypergraphs?

- Strong independent set: Pick ≤ 1 vertex from each hyperedge

  $\frac{1}{\Delta}$-approximation known [Halldòrsson, Losievskaja 2009]

- Weak independent set: Pick $\leq$ all but one vertices from each hyperedge

  $\frac{\log \Delta}{\Delta \log \log \Delta}$-approximation known [Agnarsson et al. 2013]

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Class $C \equiv$ hyperedge e, quota of $C = q(e)$

What next?

Lower quotas

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$\Omega(\ell/\log\ell)$ hardness of approximation

Proportional fairness

$$\boxed{\alpha|M(p)| \leq |M(C)| \leq \beta|M(p)|}$$

$M(p)$: Items matched to platform p
$M(C)$: Items matched to p from class C
$0 \leq \alpha \leq \beta \leq 1$

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$M(p)$: Items matched to platform p
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$O(\ell)$-approximation only for disjoint classes with slight violation of constraints

Fairness to individuals

When items have preferences, every matching is unfair to some items.

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Idea: Output a distribution on matchings
Uniform sampling from the distribution is fair to all items.

Thank you!![2]

[2]Thanks to Meghana and Nada for slides upto laminar part