Improved Sublinear-Time Algorithms for Testing Order Pattern Freeness

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- A is π-free if it has no π-appearance
- Above array has a (3,4,1,2)-appearance but is (1,2,3,4)-free

Pattern freeness

- Well studied notion in combinatorics [Bona; J. Comb. Theory '99], [Arratia; Elec. J. Of Comb. '99], [Alon & Friedgut; J. Comb. Theory '00], [Marcus & Tardos; J. Comb. Theory '04]

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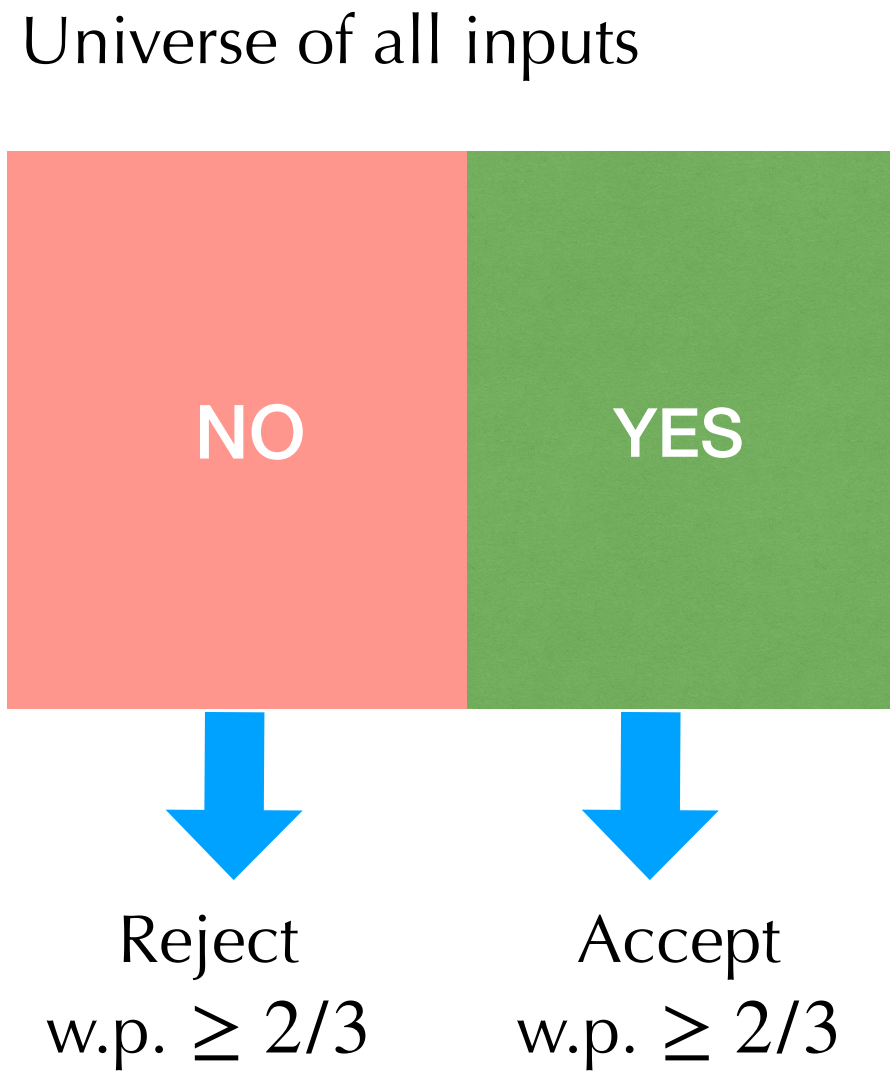
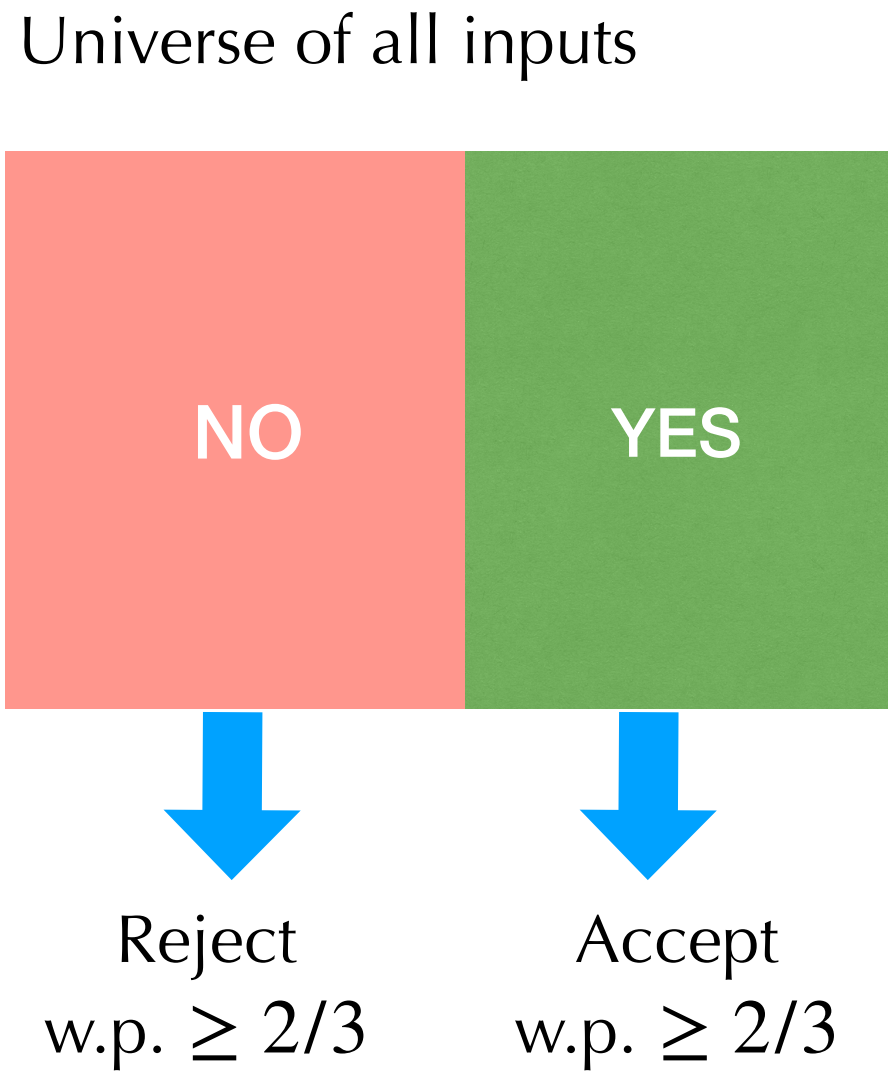
Today: Sublinear-Time Algorithms for (approximate) Pattern Freeness

Decision Problem



Decision Problem

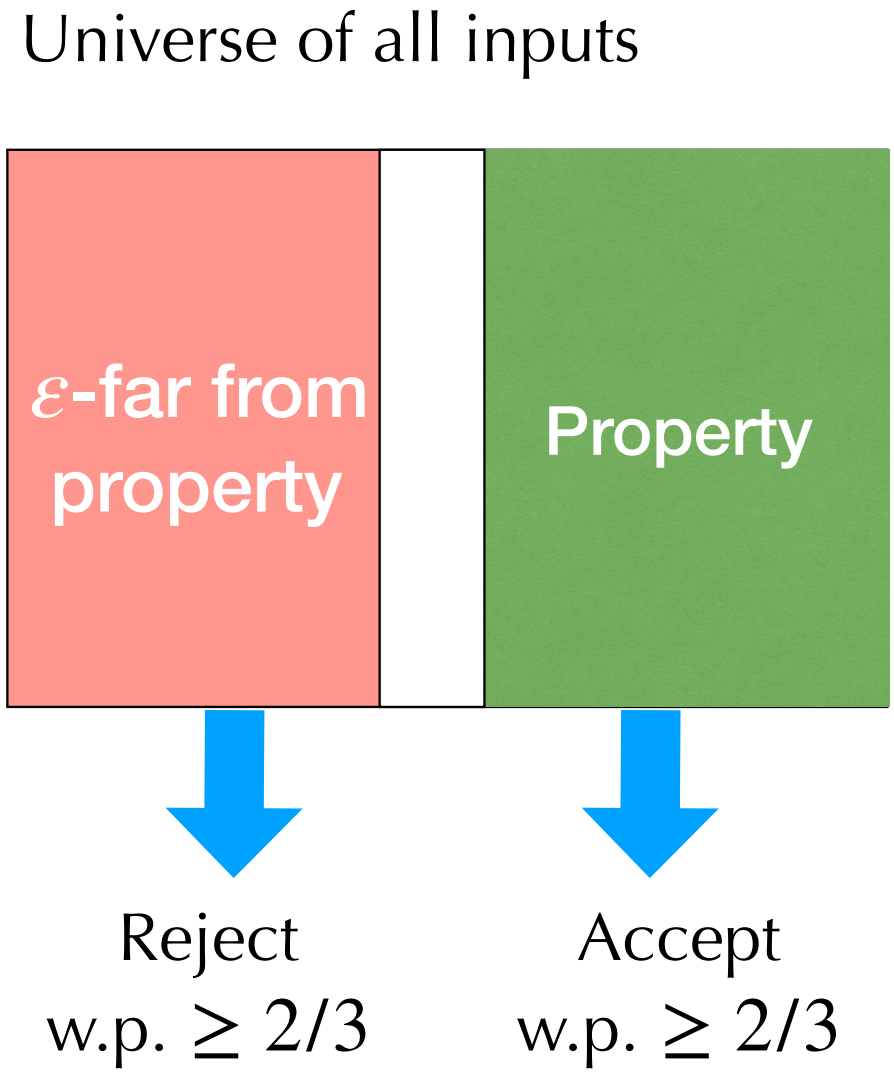
- Cannot exactly solve nontrivial decision problems without full access to the input



Property testing

[Rubinfeld & Sudan; **SODA 92 & SICOMP 96**, Goldreich, Goldwasser & Ron; **FOCS 96 & JACM 98**]

- ε-far from property: At least ε fraction of input values need to be changed to satisfy the property



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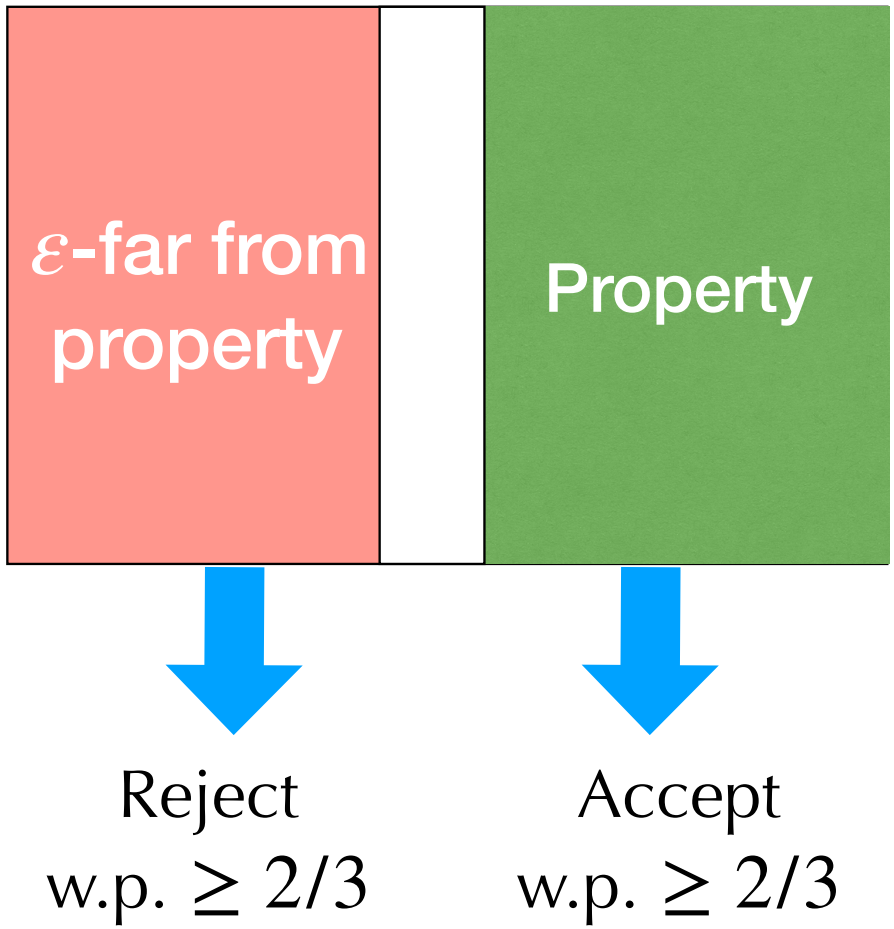

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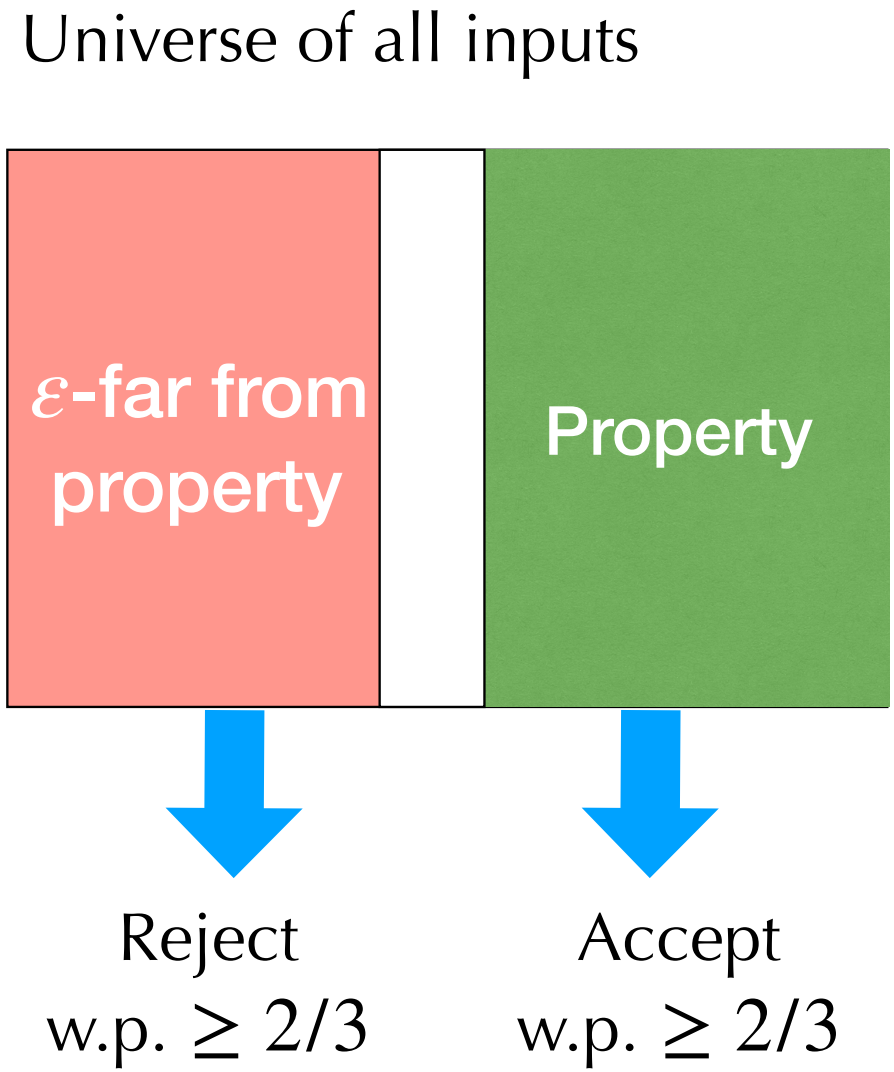
[Rubinfeld & Sudan; **SODA 92 & SICOMP 96**, Goldreich, Goldwasser & Ron; **FOCS 96 & JACM 98**]

- Several well-studied and fundamental properties
  - Bipartiteness [Alon, Krivelevich; SICOMP '02, …]
  - Monotonicity [Chakrabarty Seshadhri; STOC '13, Khot, Minzer, Safra; SICOMP '18, …]
  - Convexity of images [Berman Murzabulatov Raskhodnikova; RSA '19, …]
  - Linearity and related properties [Blum Luby Rubinfeld; JCSS '93, …]



Testing Pattern Freeness

- Given query access to an array A of length n, a parameter $\epsilon \in (0,1)$, and permutation π, decide whether
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Monotonicity $\equiv (2,1)$-freeness

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Tester adaptive if its queries depend on answers to previous queries

Tester nonadaptive otherwise

Testing Pattern Freeness

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Testing Pattern Freeness

- **Special case**: Sortedness testing of arrays of length n [Ergun Kannan Kumar Rubinfeld Vishwanathan 00, Fischer 04, …]
- [Newman Rabinovich Rajendraprasad Sohler 19] initiated the study of sublinear-time algorithms for testing pattern freeness for larger patterns
- Studied in depth for monotone patterns $(1,2,\ldots,k)$ or $(k,k-1,\ldots,1)$

  [NRRS19, BenEliezer Canonne Letzter Waingarten 19, BenEliezer Letzter Waingarten 19]

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What about adaptive testers ?

Our Result

Let $\epsilon \in (0,1), k \in \mathbb{N}, \pi \in \mathbb{S}_k$. There is an ϵ-tester for π-freeness

- with query complexity $\tilde{O}(n^{o(1)})$
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- with query complexity $\tilde{O}(n^{o(1)})$
- that always accepts π-free arrays.

- Improved sublinear-time guarantee
- Our techniques are general and do not rely on the structure of the pattern being considered (e.g., as opposed to [NRRS19])

Today

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Goal: Design an algorithm that makes $\tilde{O}(\sqrt{n})$ queries and finds a π-appearance in an array that is ϵ-far from π-free

First Useful Fact

- Array A is ϵ-far from being π-free $\Longrightarrow$ there are $\geq \frac{\epsilon n}{4}$ disjoint π-appearances

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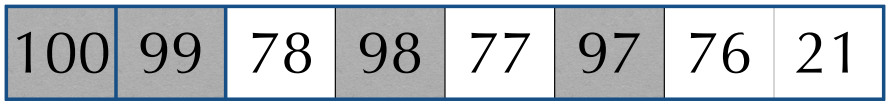
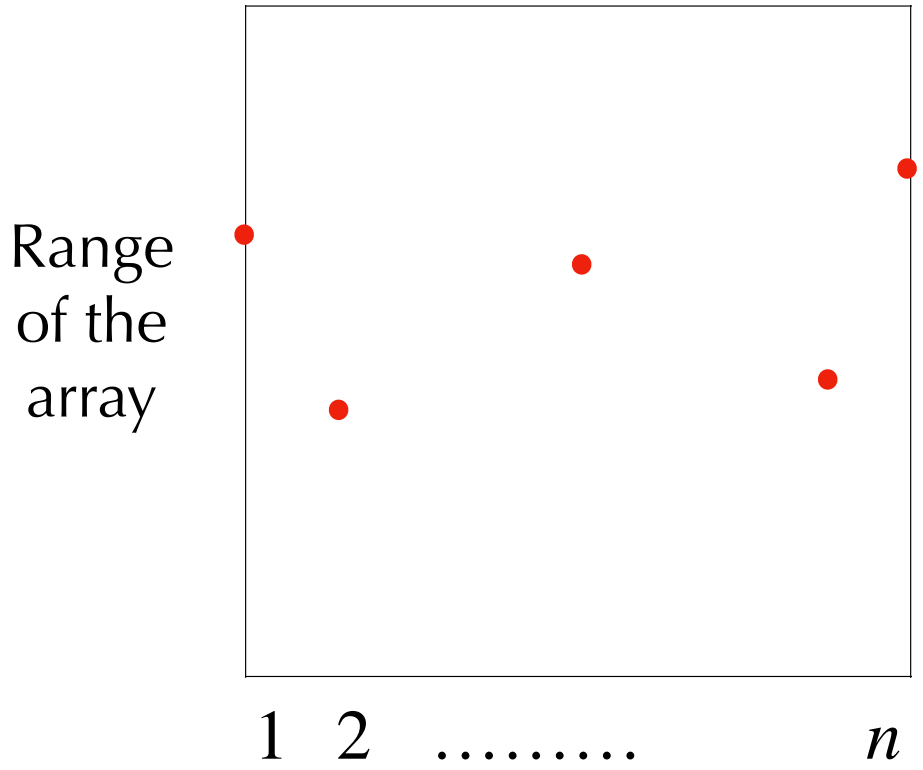


Disjoint (4,3,2,1)-appearances

Gridding

- **(Conceptual) Step 1**: View the array as a box of n points in $[n] \times \mathbb{R}$



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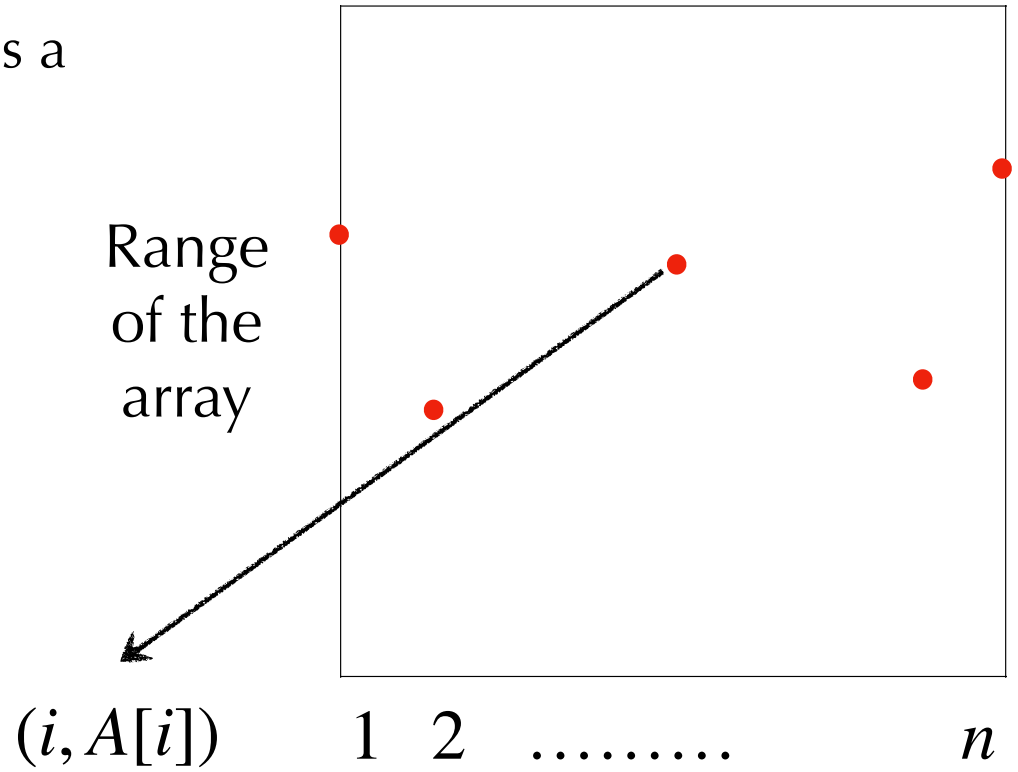

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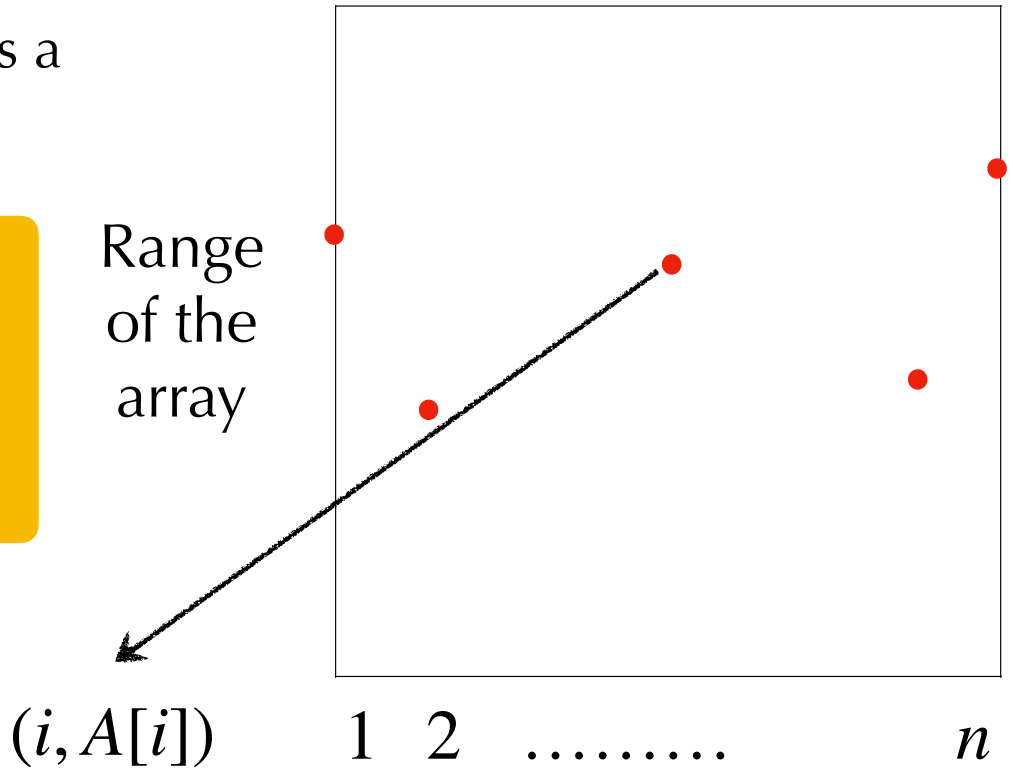

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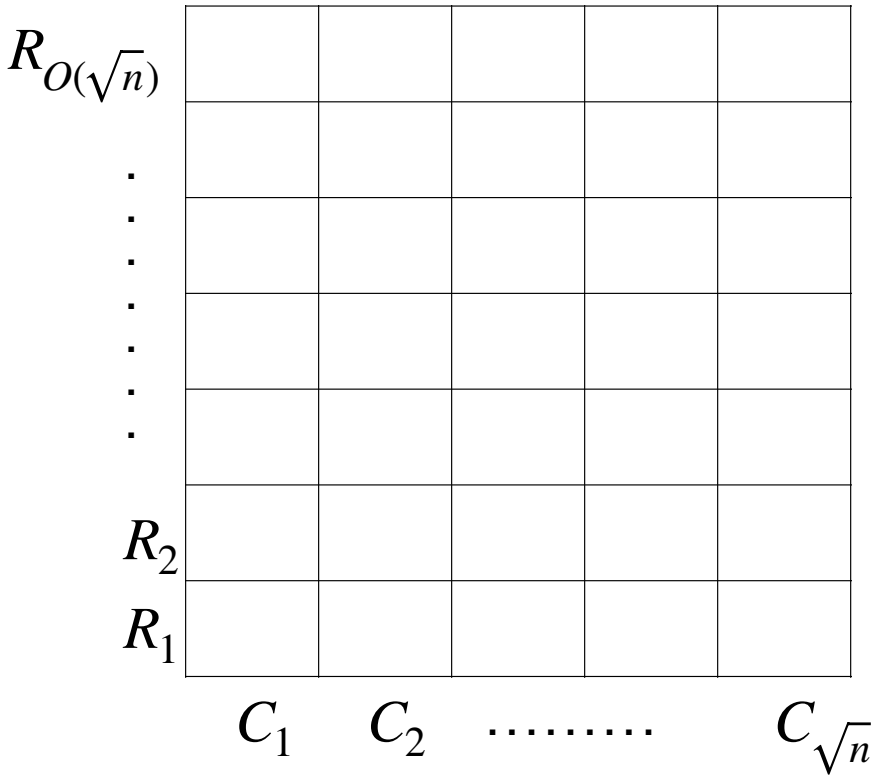

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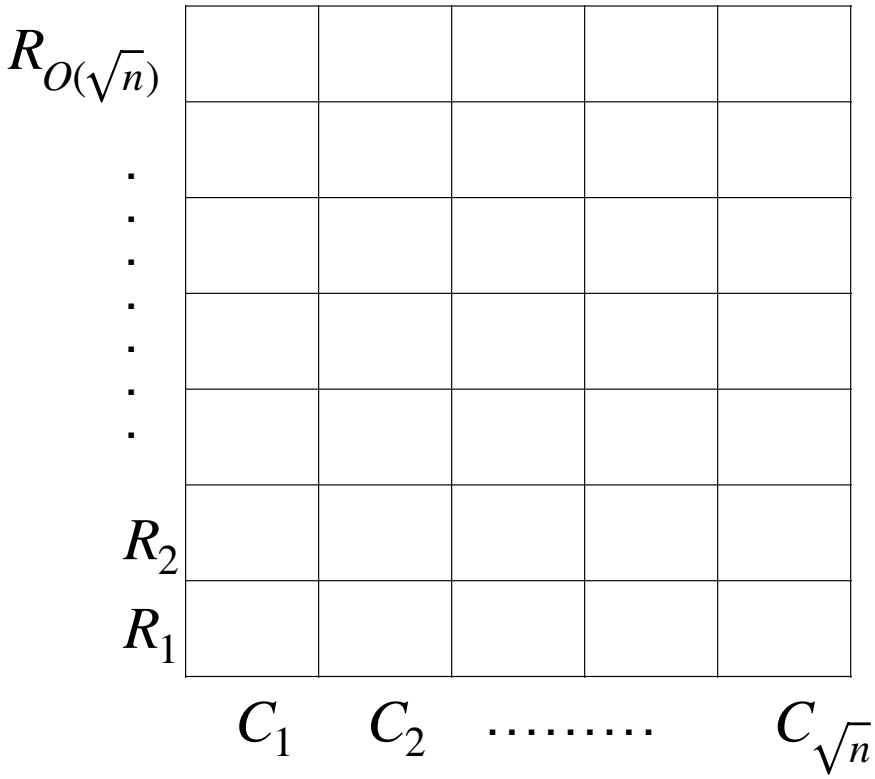
We ensure that rows and columns roughly equipartition the set of array points



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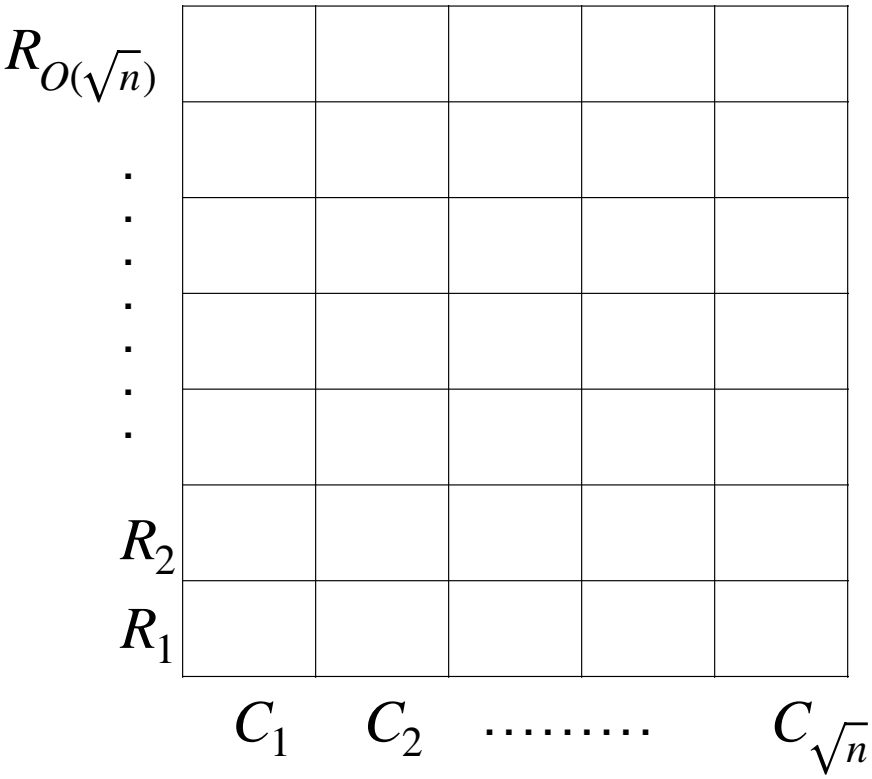
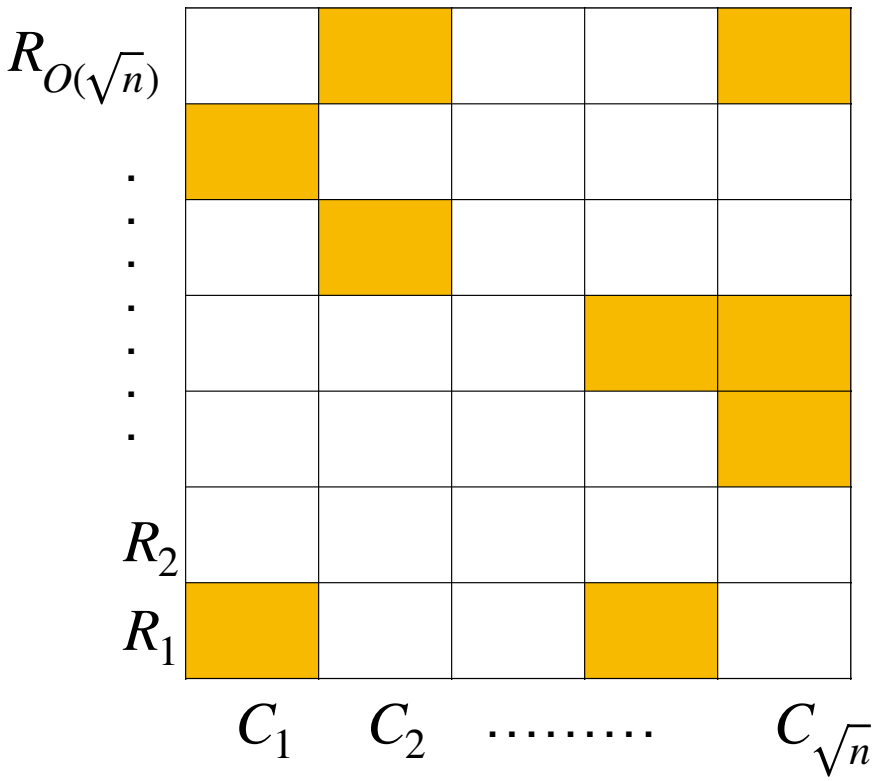
Each row has $\sim \sqrt{n}$ points
Each column has $\sqrt{n}$ points



Gridding: Part 2

- From each column, sample $\tilde{O}(1)$ uniformly random points
- Tag a box as **nonempty** if it has at least one sampled point; tag it **dense** if it has a constant fraction of sampled points
- There are $O(n)$ boxes, out of which we only tag $\tilde{O}(\sqrt{n})$ boxes



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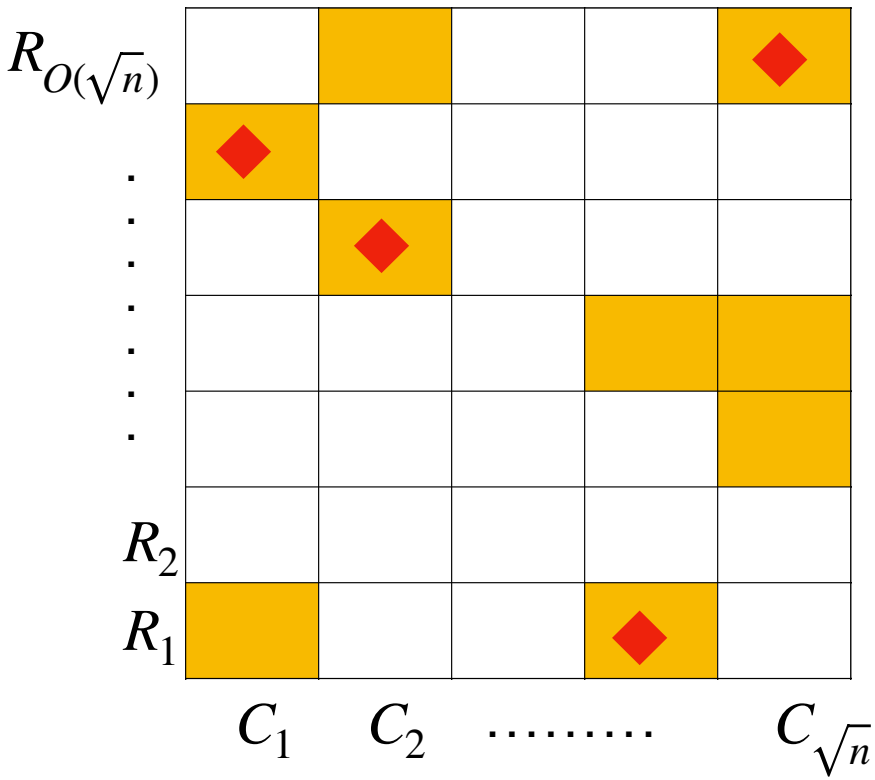
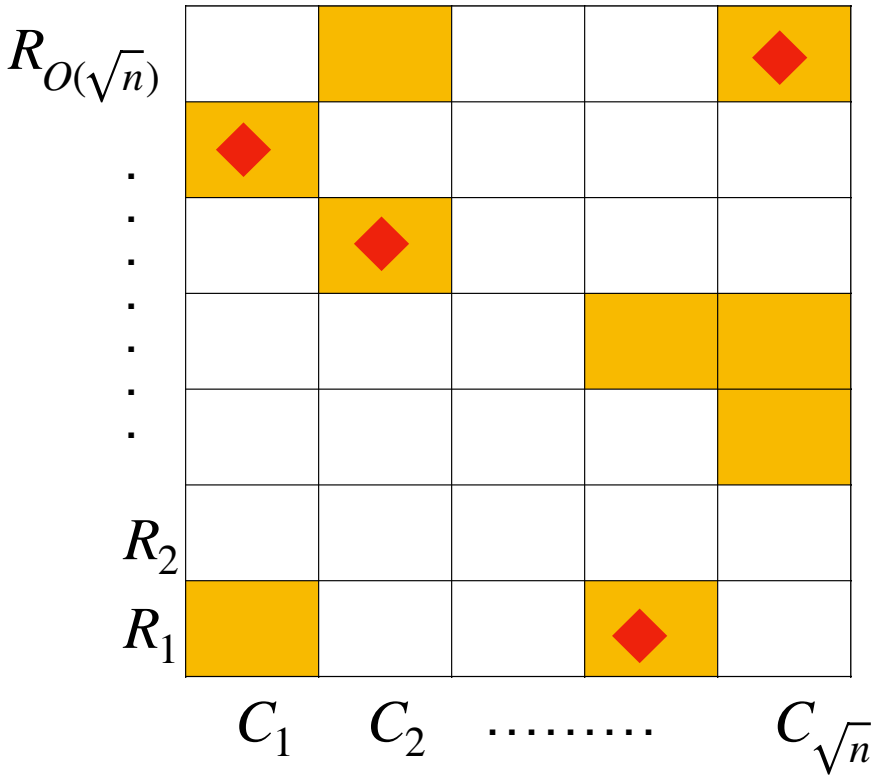
If $\pi = (3,2,1,4)$, we already found a π-appearance



In general, if too many nonempty boxes…

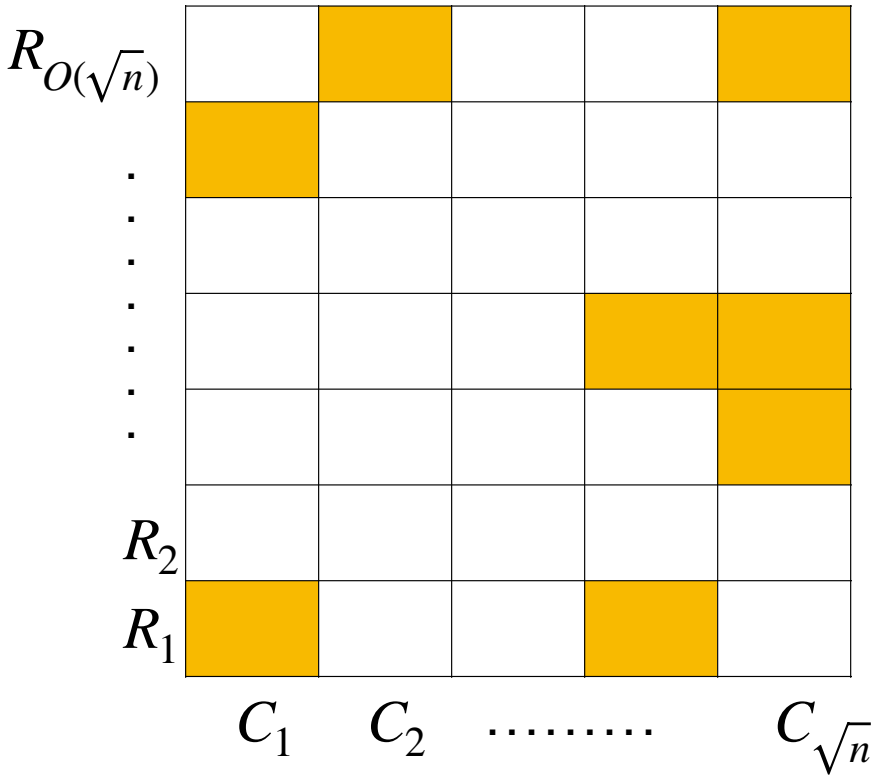
- **[Markus & Tardos '04]** For each permutation π, there exists a constant κ such that such that if the grid on the right has more than $\kappa\sqrt{n}$ nonempty boxes, then there is a π-appearance among the nonempty boxes



If gridding does not find a π-appearance

We show that:

- Only $O(\sqrt{n})$ nonempty boxes. In particular, only $O(\sqrt{n})$ dense boxes



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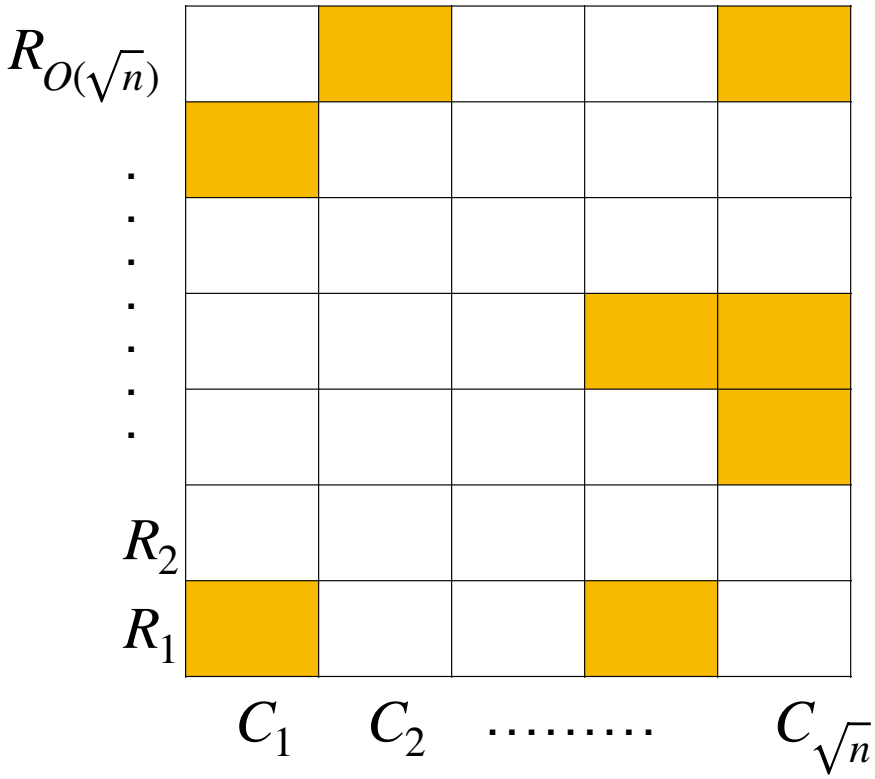

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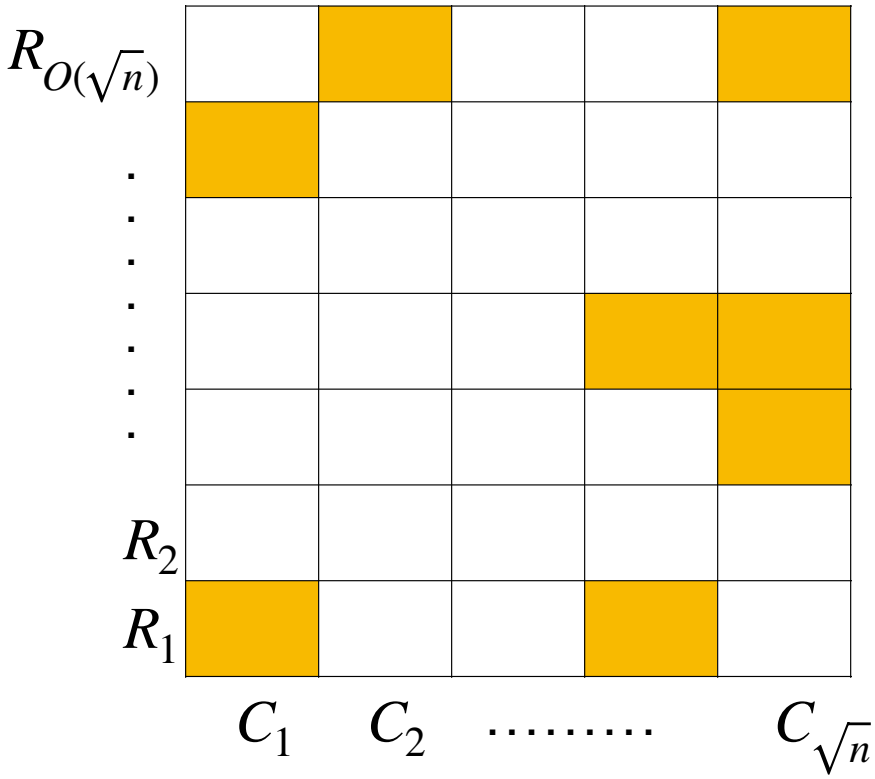

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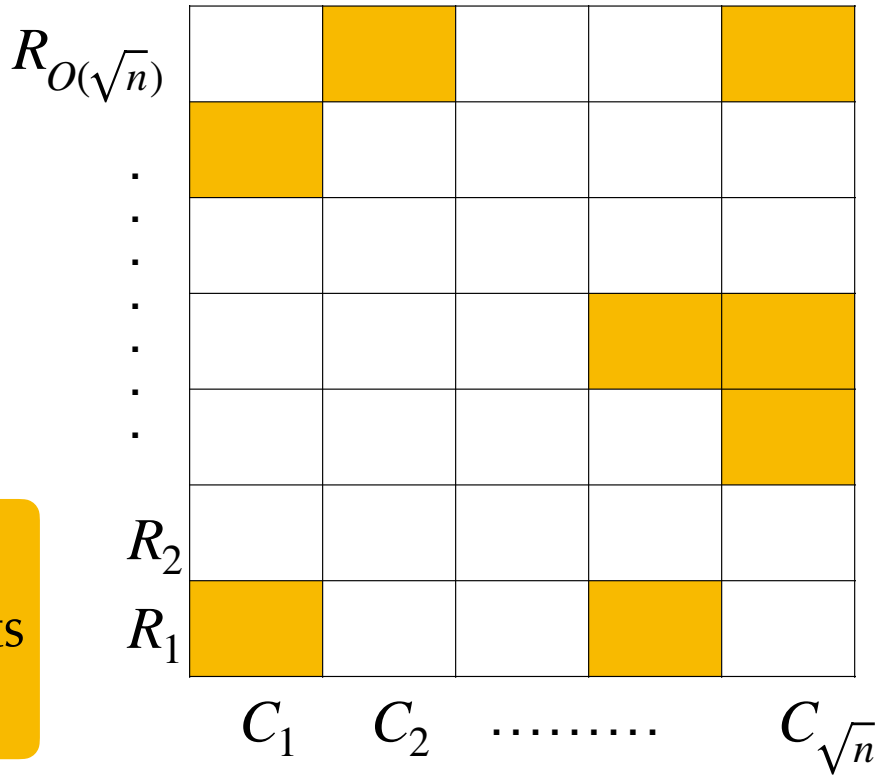
Ideas in Proof: Combinatorial lemma by [MarcusTardos04] + Averaging arguments + Properties of the sampling procedure



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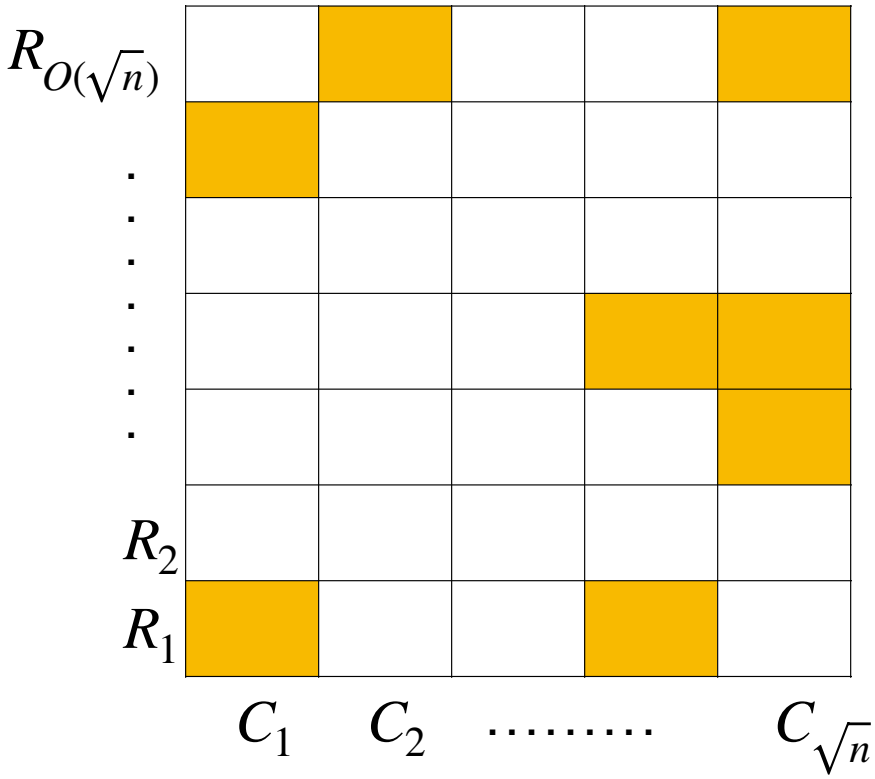
Where in the grid do these appear?



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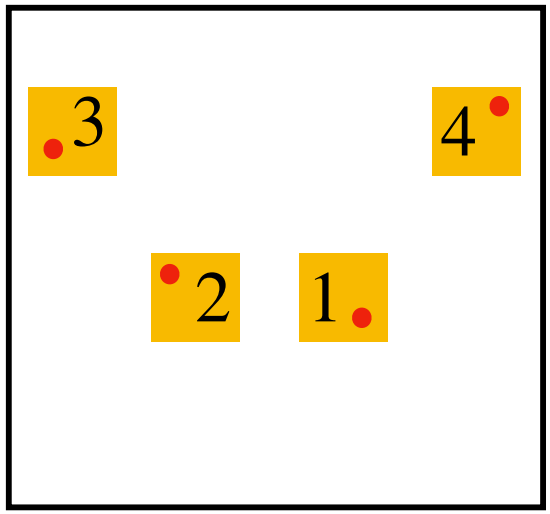

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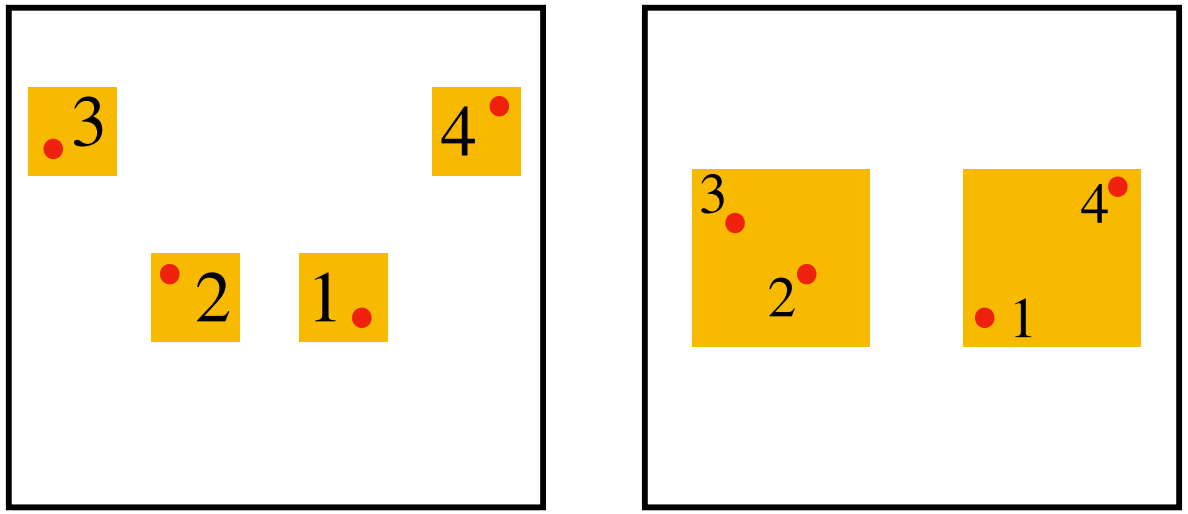

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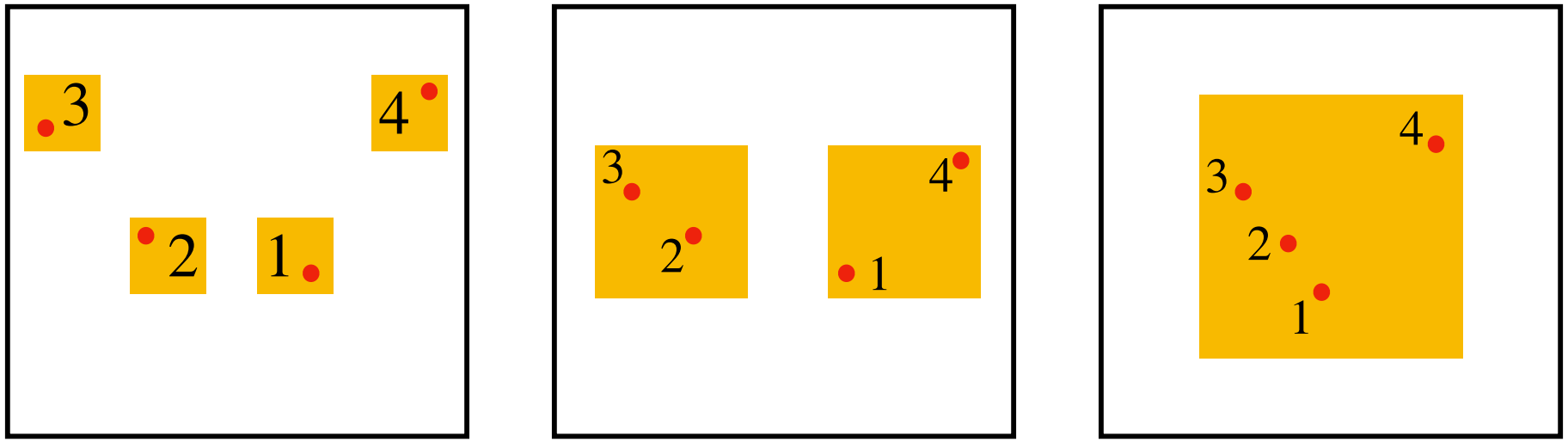

How to find π-appearances in these configurations?

- For each configuration $\mathscr{C}$:
  - Check for π-appearances whose legs are in boxes forming the configuration $\mathscr{C}$

A 2-boxed configuration

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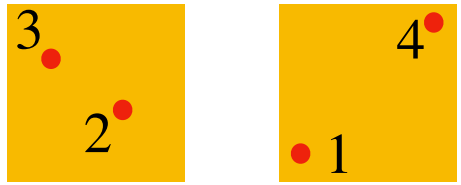

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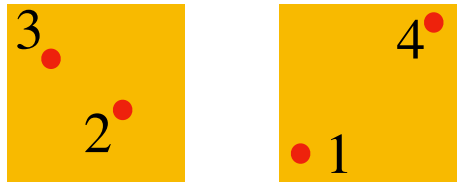

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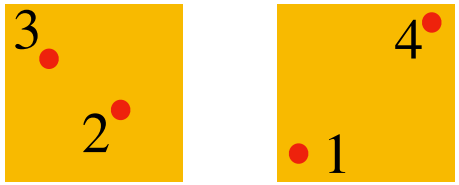

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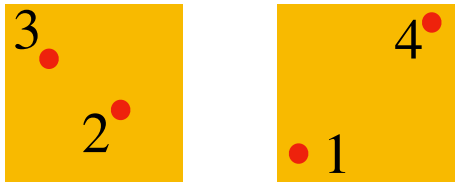

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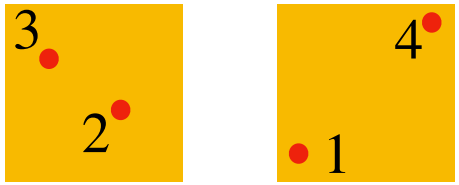

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- Querying all points in the subarrays corresponding to B and the other dense boxes in its row is sufficient to detect such a π-appearance

  $\tilde{O}(\sqrt{n})$ queries overall, since we want a high success probability

Other proof ideas

- Most other configurations are dealt with by similar averaging arguments
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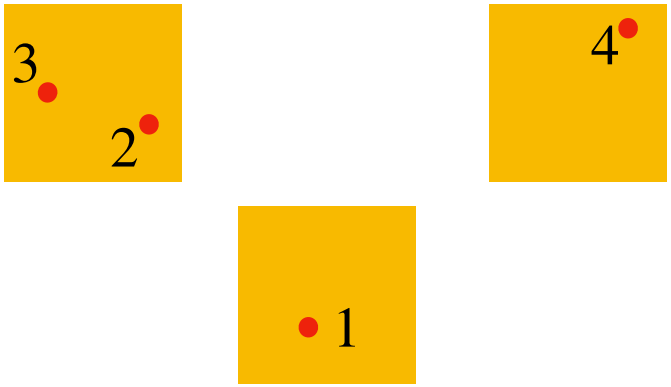

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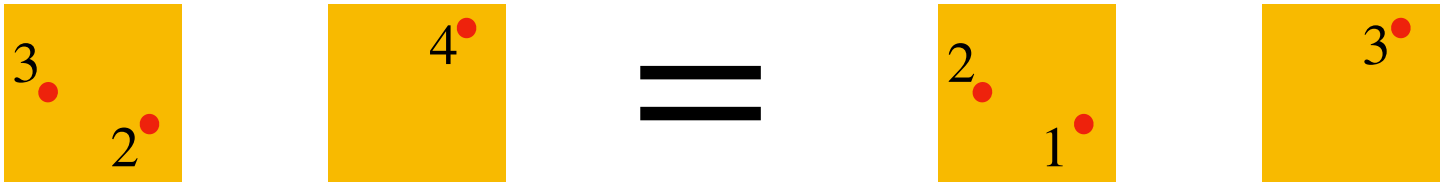

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- Our algorithm works for patterns of all constant length k
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- The algorithm is called recursively in order to improve the query complexity to $\tilde{O}(n^{o(1)})$ from $O(\sqrt{n})$

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Thank you!