On Graphs with Minimal Eternal Vertex Cover Number

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Outline

1. Introduction
2. Characterization for $\mathrm{evc}(G) = \mathrm{mvc}(G)$ for some graph classes
3. Algorithms using the characterization
4. Conclusion and Open problems

Eternal Vertex Cover (EVC) problem

- Introduced by Klostermeyer et al.[1] in 2009
- Attacker-defender game in which k guards are placed on distinct vertices of G
- In each round, attacker chooses an edge to attack
- As a response to the attack, defender has to move guards such that
  - At least one guard must move across the attacked edge.
  - Others can either remain in the current position or move to an adjacent vertex.
  - At most one guard exists on any vertex.
- If an attack cannot be defended, the attacker wins.
- The defender wins if he can defend any sequence of infinite attacks.
- Eternal vertex cover number (evc) of a graph G: The minimum number k such that the defender has a winning strategy with k guards on G.
- For any graph G, $\mathrm{mvc}(G) \leq \mathrm{evc}(G)$
- Given a graph G and an integer k, checking if $\mathrm{evc}(G) \leq k$ is NP-hard[2]

[1] William F. Klostermeyer and C. M. Mynhardt. Australas. J. Combin,2009

[2] Fedor V. Fomin, Serge Gaspers, Petr A. Golovach, Dieter Kratsch, and Saket Saurabh,Inf. Process. Lett.,2010

Eternal Vertex Cover Number (evc)-Some Examples

- $\mathrm{mvc}(P_4) = 2$ and $\mathrm{evc}(P_4) = 3$

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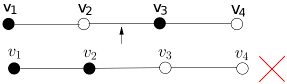

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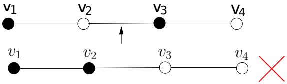

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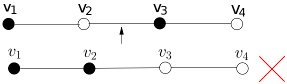

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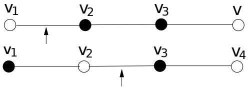

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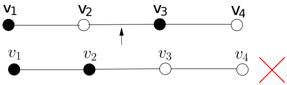

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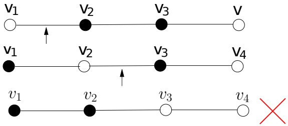

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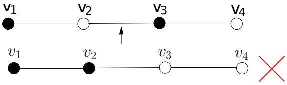

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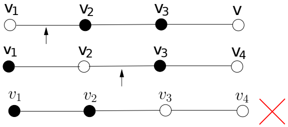

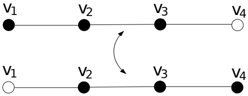

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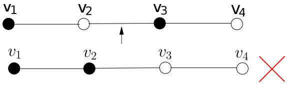

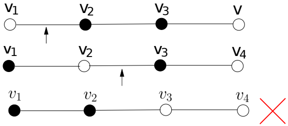

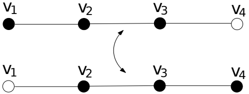

- $evc(C_n) = mvc(C_n) = \lceil \frac{n}{2} \rceil$

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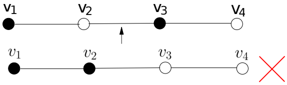

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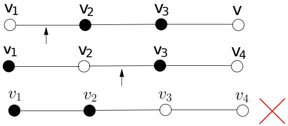

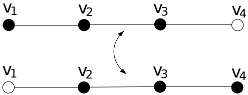

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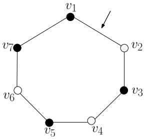

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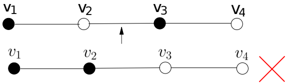

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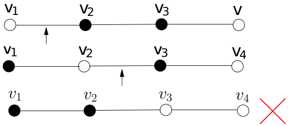

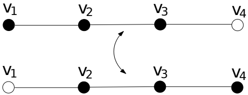

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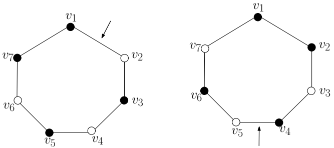

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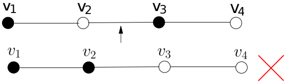

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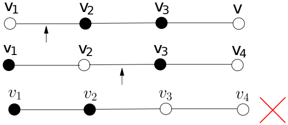

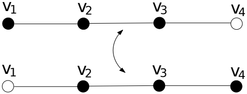

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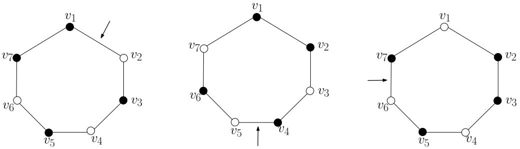

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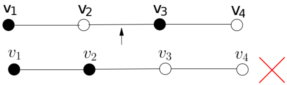

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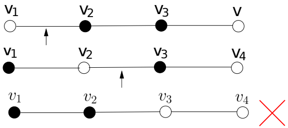

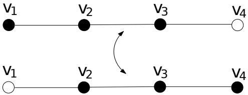

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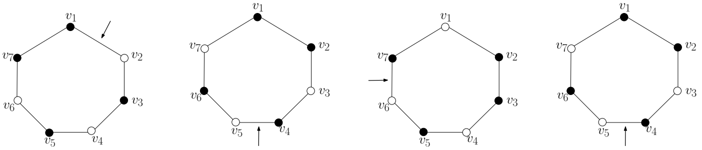

Contribution

- It is known[3] that for any graph G, $\mathrm{mvc}(G) \leq \mathrm{evc}(G) \leq 2\,\mathrm{mvc}(G)$.
- Klostermeyer et al. gave a characterization of graphs with $\mathrm{evc}(G) = 2\,\mathrm{mvc}(G)$
- Characterization of graphs with $\mathrm{evc}(G) = \mathrm{mvc}(G)$ remains open.
- We achieve such a characterization for a subclass of graphs.
- This subclass include chordal graphs and internally triangulated planar graphs.

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- This subclass include chordal graphs and internally triangulated planar graphs.

Overview of the Approach

- A simple necessary condition for $\mathrm{evc}(G) = \mathrm{mvc}(G)$ is proposed here.
- For many graph classes including chordal and internally triangulated planar graphs, the necessary condition is also shown to be sufficient.
- The characterization leads to the computation of $\mathrm{evc}(G)$ in polynomial time for some graph classes like biconnected chordal graphs.
- For some graphs including chordal graphs, if $\mathrm{mvc}(G) = \mathrm{evc}(G)$, we have a polynomial time strategy for guard movements.

[3] William F. Klostermeyer and C. M. Mynhardt. Australas. J. Combin,2009

Characterization for $\text{evc}(G) = \text{mvc}(G)$ for some graph classes

Necessary condition for any graph

If $\text{evc}(G) = \text{mvc}(G)$, then for every vertex $v \in V(G)$, $\exists$ a min VC of G containing v.

Proof:

- Suppose there are mvc guards and $\exists$ a vertex v that does not belong to any min VC of G.
- When an edge incident to v is attacked, v has to be occupied in the next configuration.
- Since there is no min VC containing v, attack cannot be handled.

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Sufficiency condition for some graph classes

- The necessary condition is also sufficient for graphs in which all min VCs are connected
- Biconnected chordal and biconnected internally triangulated graphs are some examples of such graphs.
- The characterization can be generalized for handling more graph classes.

How are connected vertex covers helpful?

- The *connected vertex cover number*, $\mathrm{cvc}(G)$, is the minimum cardinality of a connected vertex cover of G.

Lemma (Klostermeyer et al.)

Let G be a nontrivial, connected graph and D be a vertex cover of G such that $G[D]$ is connected. Then, $evc(G) \leq cvc(G) + 1 \leq |D| + 1$.

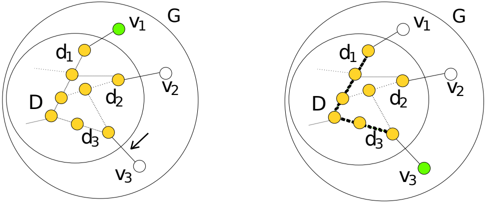


Figure: Handling attack using connected VC[4]

[4] William F. Klostermeyer and C. M. Mynhardt. Australas. J. Combin, 2009

Characterization for $\mathrm{evc}(G) = \mathrm{mvc}(G)$ for graphs with all min VCs connected

Theorem

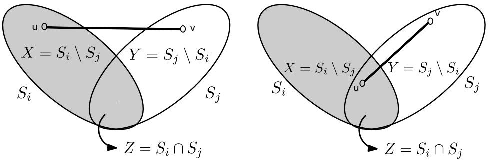
Let $G(V, E)$ be a connected graph with $|V| \geq 2$ such that every min VC of G is connected. Then $\mathrm{evc}(G) = \mathrm{mvc}(G)$ if and only if for every vertex $v \in V$, there exists a min VC of G containing v.

Proof:

$\Longrightarrow$ Trivial from necessary condition

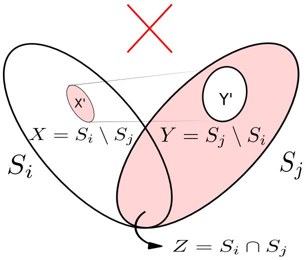
$\Longleftarrow$ **Claim 1:** For any min VC S_i of G, an attack on any edge uv with $u \in S_i$ and $v \notin S_i$ can be defended by moving to a min VC S_j such that $v \in S_j$ and $|S_i \triangle S_j|$ is minimum.

- X and Y are independent sets
- $H = G[X \uplus Y]$ **is a bipartite graph**
- Since $|S_i| = |S_j|$, $|X| = |Y|$



Proof of Claim 1

Claim 1.1: $H = G[X \uplus Y]$ has a perfect matching.
(**Recall:** $H = G[X \uplus Y]$ is a bipartite graph),



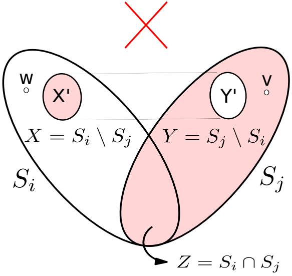
Proof strategy:

- Consider $Y' \subseteq Y$
- $X' = N_H(Y')$
- Suppose $|X'| < |Y'|$.
- Let $S' = Z \uplus (Y \setminus Y') \uplus X'$
- $|S'| < \text{mvc}(G)$. $\Rightarrow\Leftarrow$

$\forall Y' \subseteq Y$, $|N_H(Y')| \geq |Y'|$ and by Hall's theorem H has a perfect matching.

Proof of Claim 1...

Claim 1.2: $\forall w \in X$, **the bipartite graph** $H \setminus \{w, v\}$ **has a perfect matching.**
(**Recall:** S_j is a min VC such that $v \in S_j$ and $|S_i \triangle S_j|$ is minimum)



Proof strategy:

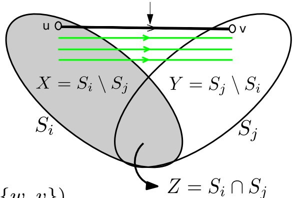
- $Y' \subseteq (Y \setminus \{v\})$
- $|X'| = |N_H(Y')|$
- By Claim 1.1, $|X'| \geq |Y'|$.
- Suppose $|X'| = |Y'|$.
- Let $S' = Z \uplus (Y \setminus Y') \uplus X'$
- $|S' \triangle S_i| < |S_j \triangle S_i|$. $\Rightarrow\Leftarrow$
- Therefore, $|X'| > |Y'|$

$\forall Y' \subseteq (Y \setminus \{v\})$, $|N_H(Y') \setminus \{w\}| \geq |Y'|$ and by Hall's theorem, $H \setminus \{w, v\}$ has a perfect matching.

Handling attack on uv by moving to S_j

Claim 1: For any min VC S_i of G, an attack on any edge uv with $u \in S_i$ and $v \notin S_i$ can be defended by moving to a min VC S_j such that $v \in S_j$ and $|S_i \triangle S_j|$ is minimum.

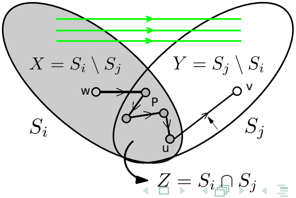
1. $u \in X$: (Using perfect matching M in $H \setminus \{u, v\}$)



2. $u \notin X$: (Using perfect matching M in $H \setminus \{w, v\}$)

Connectivity of S_i is crucial here

- w : nearest vertex of u in X
- P : shortest path from u to w in S_i



Deciding evc(G) when all min VCs are connected

Theorem

Let $G(V, E)$ be a graph for which every min VC *is connected. If for every vertex $v \in V$, there exists a min* VC *S_v of G such that $v \in S_v$, then* $\mathrm{evc}(G) = \mathrm{mvc}(G)$. *Otherwise,* $\mathrm{evc}(G) = \mathrm{mvc}(G) + 1$.

- The second case follows from $\mathrm{evc}(G) \leq cvc(G) + 1$

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- The second case follows from $\text{evc}(G) \leq cvc(G) + 1$

Consequence:

- If all min VCs of G are connected, then deciding $\text{evc}(G) \leq k$ is in NP.
- For biconnected chordal graphs and biconnected internally triangulated graphs, all min VCs are connected and hence deciding $\text{evc}(G) \leq k$ is in NP.
- If all min VCs of G are connected and the necessary condition can be checked in polynomial time, then $\text{evc}(G)$ can be computed in polynomial time.
- For biconnected chordal graphs, $\text{evc}(G)$ can be computed in polynomial time.

Generalization of the characterization

Necessary condition

Let $G(V, E)$ be any connected graph. Let $X \subseteq V$ be the set of cut vertices of G. If $\text{evc}(G) = \text{mvc}(G)$, then for every vertex $v \in V \setminus X$, there exists a min VC S_v of G such that $(X \cup \{v\}) \subseteq S_v$.

proof idea:

- All vertices of X have to be occupied in all configurations.
- When an edge incident to v is attacked, $(X \cup \{v\})$ has to be occupied.

Sufficiency condition for some class of graphs

Let $G(V, E)$ be a connected graph with $|V| \geq 2$ and $X \subseteq V$ be the set of cut vertices of G. Suppose every min VC S of G with $X \subseteq S$ is connected. If for every vertex $v \in V \setminus X$, there exists a min VC S_v of G such that $(X \cup \{v\}) \subseteq S_v$, then $\text{evc}(G) = \text{mvc}(G)$.

Polynomial time algorithms

- A class of graphs $\mathcal{H}$ is called hereditary, if deletion of vertices from any graph G in $\mathcal{H}$ would always yield another graph in $\mathcal{H}$.
- Chordal graphs form a hereditary graph class.

Theorem

If $\mathcal{H}$ is a hereditary graph class such that:

- *for every graph G in $\mathcal{H}$, $\mathrm{mvc}(G)$ can be computed in polynomial time and*
- *for every biconnected graph H in $\mathcal{H}$, all vertex covers of H are connected.*

Then,

1. *for any graph G in $\mathcal{H}$, in polynomial time we can decide whether* $\mathrm{evc}(G) = \mathrm{mvc}(G)$
2. *for any graph G in $\mathcal{H}$ with $\mathrm{evc}(G) = \mathrm{mvc}(G)$, there is a polynomial time strategy for guard movements using $\mathrm{evc}(G)$ guards.*
3. *for any biconnected graph G in $\mathcal{H}$, in polynomial time we can compute $\mathrm{evc}(G)$. Moreover, there is a polynomial time strategy for guard movements using $\mathrm{evc}(G)$ guards.*

Polynomial time strategy for guard movements

Corollary

For any chordal graph G, we can decide in polynomial-time whether $\mathrm{evc}(G) = \mathrm{mvc}(G)$. *Also, if* $\mathrm{mvc}(G) = \mathrm{evc}(G)$, *there is a polynomial-time strategy for guard movements using* $\mathrm{evc}(G)$ *guards.*

Corollary

If G is a biconnected chordal graph, then we can determine $\mathrm{evc}(G)$ *in polynomial-time. Moreover, there is a polynomial-time strategy for guard movements using* $\mathrm{evc}(G)$ *guards.*

Conclusion and Open problems

- In certain graph classes, we gave a condition for characterizing graphs with $\mathrm{evc}(G) = \mathrm{mvc}(G)$.
- The characterization does not hold for biconnected bipartite planar graphs.

- Obtaining a characterization for bipartite graphs is an interesting open problem.
- Identify other graph classes for which this characterization holds.

Thank You !