Indian Institute of Technology Jodhpur
CS112 Discrete Mathematics

Maximum Marks 20 Time:1 Hour

1. Find the coefficient of x^{52} in $(x^{10}+x^{11}+\ldots+x^{25})(x+x^2+\ldots+x^{15})(x^{20}+x^{21}+\ldots+x^{45})$[5 marks]

   $(x^{10}+x^{11}+\ldots+x^{25})(x+x^2+\ldots+x^{15})(x^{20}+x^{21}+\ldots+x^{45})$
   $=x^{31}\cdot\frac{1-x^{16}}{1-x}\cdot\frac{1-x^{15}}{1-x}\cdot\frac{1-x^{26}}{1-x}$
   $=x^{31}\cdot(1-x^{16})\cdot(1-x^{15})\cdot(1-x^{26})\cdot(1-x)^{-3}$
   $=x^{31}\cdot(1-x^{16})\cdot(1-x^{15})\cdot(1-x^{26})\cdot(1+\binom{3}{1}x+\binom{4}{2}x^2+\cdots)$
   Number of ways
   $52=31+16+5$
   $52=31+15+6$
   $52=31+21$

   $-\binom{5+3-1}{5}-\binom{6+3-1}{6}+\binom{21+3-1}{21}=204$

2. Let G be a graph (not necessarily bipartite) on $2n$ vertices, such that all degrees are at least n. Show that G has a perfect matching. [10 marks]

   Let $e_1,\cdots,e_k$ be a matching of maximal size. Suppose $k<n$ and let u and v be two unmatched vertices. There are at least $2n$ edges between $\{u,v\}$ and the $2k$ matched vertices. Thus, there is an edge $e_i=ab$ such that there are at least 3 edges between $\{a,b\}$ and $\{u,v\}$. W.l.o.g. let those edges be au,bu,av. Then by replacing e_i in the matching by av and bu we get a larger matching. Contradiction.

3. Show that a maximal matching is at least half the size of a maximum matching [5 marks]

   Let M be a maximal matching and N a maximum matching. Suppose $|M|<|N|/2$. Then the number of vertices in edges of M is strictly less than the number of edges in N, so there must be an edge $e\in N$ with neither endpoint in an edge of M. But then we could add e to M, contrary to it being maximal.

4. Take a standard deck of cards, and deal them out into 13 piles of 4 cards each. Show that it is always possible to select exactly 1 card from each pile, such that the 13 selected cards contain exactly one card of each rank (ace, $2,3,\cdots$, queen, king).

   Consider a bipartite graph $G[X,Y]$, where X is the set of 14 piles and Y is the set of 14 possible ranks, and each pile is connected by an edge with the ranks that appear in it. Clearly, for any k piles, there are at least k ranks appearing in them. Thus, the marriage condition is satisfied, and therefore we have a perfect matching.

5. Suppose that we are given a sequence of $nm+1$ distinct real numbers. Prove that there is an increasing subsequence of length $n+1$ or a decreasing subsequence of length $m+1$. [7 marks]

   Let the sequence be $a_1, ..., a_{nm}+1$. For each $k, 1 \leq k \leq nm+1$, let $f(k)$ be the length of the longest decreasing subsequence that starts with a_k, and let $g(k)$ be the length of the longest increasing subsequence that starts with a_k. Notice that $f(k), g(k) \geq 1$ always. If there is a k with either $f(k) \geq n+1$ or $g(k) \geq m+1$, we are done. If not, then for every k we have $1 \leq f(k) \leq n$ and $1 \leq g(k) \leq m$. Set up nm pigeonholes, with each pigeonhole labeled by a different pair $(i, j), 1 \leq i \leq n, 1 \leq j \leq m$ (there are exactly nm such pairs). For each $k, 1 \leq k \leq nm+1$, put ak in pigeonhole (i, j) iff $f(k) = i$ and $g(k) = j$. There are nm+ 1 pigeonholes, so one pigeonhole, say hole (r, s), has at least two pigeons in it. In other words, there are two terms of the sequence, say a_p and a_q (where without loss of generality $p < q$), with $f(p) = f(q) = r$ and $g(p) = g(q) = s$. Suppose $a_p \geq a_q$. Then we can find a decreasing subsequence of length r + 1 starting from a_p, by starting a_p, a_q, and then proceeding with any decreasing subsequence of length r that starts with a_q (one such exists, since f(q) = r). But that says that f(p) r + 1, contradicting f(p) = r. On the other hand, suppose $a_p \leq a_q$. Then we can find an increasing subsequence of length s + 1 starting from a_p, by starting a_p, a_q, and then proceeding with any increasing subsequence of length s that starts with a_q (one such exists, since g(q) = s). But that says that g(p) s + 1, contradicting g(p) = s. So, whether $a_p \geq a_q$ or $a_p \leq a_q$, we get a contradiction, and we CANNOT ever be in the case where there is NO k with either f(k) n + 1 or g(k) m + 1. This completes the proof.