Indian Institute of Technology Jodhpur
CS112 Discrete Mathematics

Maximum Marks 20 Time:1 Hour

1. How many different ways can we choose 12 deserts if 5 different varieties are available? [5 marks]

   $C(5+12-1,12) = 1820$

2. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 5$ where each x is a non negative integer such that $x_1 \in [0,3]$, $x_2 \in [0,2]$ and $x_3, x_4, x_5 \geq 0$ [10 marks]

   Coefficient of z^5 in the product:

   $(1+z+z^2+z^3)(1+z+z^2)(1+z+z^2+\ldots)(1+z+z^2+\ldots)(1+z+z^2+\ldots)$

   $=\frac{(1-z^4)(1-z^3)}{(1-z)^5} = (1-z^3-z^4+z^7)\cdot\sum_{n\geq 0}\binom{n+4}{4}z^n$

   The coefficient of z^5 is $\binom{5+4}{4} - \binom{2+4}{4} - \binom{1+4}{4}$

3. Given n letters and n addressed envelopes, in how many ways can the letters be placed in the envelopes so that no letter is in the correct envelope? [5 marks]
   We want to count D_n, the number of derangements of $1,\ldots,n$. Let T_i be the set of permutations which leave i in its natural position. Then $D_n = |T_1^c \cap \ldots T_n^c| = \sum_{1\leq i\leq n}(-1)^{k+1}\frac{n!}{i!}$

4. Solve $a_n = a_{n-1} + 6a_{n-2}, a_0 = 3, a_1 = 6$ [5 marks]

   $x^n = x^{n-1} + 6x^{n-2}$

   $x^2 - x - 6 = 0$

   $x = 3, -2$

   $a_n = 3^n b_1 + (-2)^n b_2$

   $b_1 = \frac{12}{5}, b_2 = \frac{3}{5}$

   $a_n = 3^n\frac{12}{5} + (-2)^n\frac{3}{5}$

5. In a group of n people, one person may have 0 or more friends. Show that there are person who have an identical number of friends within the group. [5 marks]

   The maximum number of friends one person in the group can have is n-1, and the minimum is 0. If all of the members have at least one friend, then each individual can have somewhere between 1 to $n-1$ friends; as there are n individuals, by pigeonhole there must be at least two with the same number of friends. If one individual has no friends, then the remaining friends must have from 1 to $n-2$ friends for the remaining friends not to also have no friends. By pigeonhole again, this leaves at least 1 other person with 0 friends.