Error?

Theory:
\[ g = 9.8 \text{ m/sec}^2 \]

Measured:
\[ g = 9.7 \text{ m/sec}^2 \]

Relative error
\[ = \frac{9.7 - 9.8}{9.7} = -0.01 \]

Measurement Error?

(how well do you know your measurement of \( g = 9.7 \text{ m/sec}^2 \). Do you know it exactly 9.7 m/sec\(^2\) ? )
Accuracy and Precision

The accuracy of a measurement system is the degree of closeness of measurements of a quantity to its true (actual) value.

The precision of a measurement system, also called reproducibility, is the degree to which repeated measurements under unchanged conditions show the same results.
What is the weight?

Least count = 0.1 kg

Weight = 50.8 \pm 0.05 \text{ kg}
Errors in measurement

All digital measuring devices have a maximum uncertainty of the order of half its last digit. i.e. uncertainty is 0.005 Volts. Thus, assuming the voltmeter is calibrated accurately, the value is 12.880 ± 0.005 V.

Length of pencil is close to 36 mm. Best estimate = 36.0 mm
Probable range 35.5 to 36.5 mm
Therefore, \( l = 36.0 \pm 0.5 \text{ mm} \).
Importance of uncertainty

Density of gold = 15.5 gm/cm³
Density of alloy = 13.8 gm/cm³

Measurement by Dhoni and Kohli

DHONI best estimate is 15; probable range: 13.5 to 16.5 gm/cm³

KOHLI best estimate is 13.9 and probable range: 13.7 to 14.1 gm/cm³
Types of experimental errors

• **Systematic error**: A clock running consistently 5% late. Hard to detect. Errors of this type affect all measurements in same way. They may result from faulty calibration or bias on part of the observer.

• **Random Error**: Fluctuation in observations. These errors can be reduced only by repeated measurements. Reducing them is essentially one of improving experimental method and refining techniques.
Precise with systematic error

- Measured data
- True values
Precise with ?
Accurate with?
What is the total weight?

Total weight =
weight left side (p) + weight right side (q)

Total weight = (31.10±0.05) + (31.30±0.05) kg
Maximum probable error in difference and sum

(Measured $p$) = $p_{\text{best}}$ $\delta p$

(Measured $q$) = $q_{\text{best}}$ $\delta q$

Best estimate for ($p+q$) is ($p_{\text{best}}$ + $q_{\text{best}}$)

Highest probable value = ($p_{\text{best}}$ + $q_{\text{best}}$) + ($\delta p$ + $\delta q$)

Lowest probable value = ($p_{\text{best}}$ + $q_{\text{best}}$) - ($\delta p$ + $\delta q$)

Uncertainty in the difference ($p-q$) is also equal to the sum ($\delta p$ + $\delta q$) of the original uncertainties.
Fractional or Relative uncertainty

\[
\text{Fractional uncertainty} = \frac{\delta x}{|x_{\text{best}}|}
\]

Length \( l = 50 \ 1 \text{ cm} \)

\[
\frac{\delta l}{|l_{\text{best}}|} = \frac{1 \text{ cm}}{50 \text{ cm}} = 0.02
\]
Relative errors of products and ratio of two variables

\[
\delta \left( \frac{x}{y} \right) = \frac{\delta x}{x_{best}} + \frac{\delta y}{y_{best}}
\]

\[
\frac{x}{y} = \frac{x_{best} \pm \delta x}{y_{best} \pm \delta y} \approx \frac{x_{best}}{y_{best}} \cdot \frac{1 \pm \delta x}{1 \pm \delta y}
\]

\[
\max = \frac{x_{best}}{y_{best}} \cdot \frac{1 + \frac{\delta x}{x_{best}}}{1 - \frac{\delta y}{y_{best}}} \approx \frac{x_{best}}{y_{best}} \left( 1 + \frac{\delta x}{x_{best}} + \frac{\delta y}{y_{best}} \right)
\]

\[
\min = \frac{x_{best}}{y_{best}} \cdot \frac{1 - \frac{\delta x}{x_{best}}}{1 + \frac{\delta y}{y_{best}}} \approx \frac{x_{best}}{y_{best}} \left( 1 - \frac{\delta x}{x_{best}} - \frac{\delta y}{y_{best}} \right)
\]
Summary: Two Simple Rules

When the measured quantities are added or subtracted, the errors add.

When the measured quantities are multiplied or divided, the relative errors add.
Why to improve addition rule?

\[ x = 5.3 + 0.2 \text{ cm} \]
\[ y = 7.5 - 0.2 \text{ cm} \]
\[ y = 7.5 + 0.2 \text{ cm} \]
\[ \delta x = 0.2 \text{ cm}, \text{ probability of } \delta y \text{ is 0.2 is 0.5} \]
Propagation of errors

We have upper bounds on errors for sum/difference and product/quotient of 2 measurables. Can we do any better?

If errors are independent and random: the errors are *added in quadrature*.

\[
q = x + y
\]

\[
\delta q = \sqrt{(\delta x)^2 + (\delta y)^2} \leq \delta x + \delta y
\]

\[
q = x + \ldots + z - (u + \ldots + w)
\]

\[
\delta q = \sqrt{(\delta x)^2 + \ldots + (\delta z)^2 + (\delta u)^2 + \ldots + (\delta w)^2} \leq \\
\leq \delta x + \ldots + \delta z + \delta u + \ldots + \delta w
\]
Propagating errors

\[ l_1 = 5.3 \pm 0.2 \text{cm} \]
\[ l_2 = 7.2 \pm 0.2 \text{cm} \]
\[ l = l_1 + l_2 \]
\[ \delta l = \sqrt{ (\delta l_1)^2 + (\delta l_2)^2 } = \sqrt{ (0.2)^2 + (0.2)^2 } \approx 3 \text{mm} \]
\[ = \delta l_1 + \delta l_2 = 2 \text{mm} + 2 \text{mm} = 4 \text{mm} \]

For 2 measurables there is no great difference, but for \( n \) measurables the difference is \( \frac{1}{\sqrt{n}} \).
Propagation of errors

Relative error of product/quotient:

\[ q = \frac{x \times \ldots \times z}{u \times \ldots \times w} \]

\[ \frac{\delta q}{|q|} = \sqrt{\left( \frac{\delta x}{x} \right)^2 + \ldots + \left( \frac{\delta z}{z} \right)^2 + \left( \frac{\delta u}{u} \right)^2 + \ldots + \left( \frac{\delta w}{w} \right)^2} \]

\[ \leq \frac{\delta x}{x} + \ldots + \frac{\delta z}{z} + \frac{\delta u}{u} + \ldots + \frac{\delta w}{w} \]

If the relative errors for n measurables are the same, we gain \( \frac{1}{\sqrt{n}} \) in relative error.
General formula for propagation of error:

If \( q = q(x_i, x_{i+1}, \ldots, x_n) \), then

\[
\delta q = \sqrt{\left( \frac{\partial q}{\partial x_i} \delta x_i \right)^2 + \cdots + \left( \frac{\partial q}{\partial x_n} \delta x_n \right)^2}
\]

(Provided all errors are independent and random)

For \( q = x^n \) fractional uncertainty is \( \frac{\delta q}{|q|} = n \frac{\delta x}{|x|} \)
How to use general formula?

\[ g(L, T) = 4\pi^2 \frac{L}{T^2} \]

\[ \delta g = \sqrt{\left(\frac{\partial g}{\partial L} \delta L\right)^2 + \left(\frac{\partial g}{\partial T} \delta T\right)^2} \]

\[ \frac{\partial g}{\partial L} = 4\pi^2 \frac{1}{T^2} \]

\[ \frac{\partial g}{\partial T} = -4\pi^2 \frac{L}{T^3} \]
Propagation of errors

Example: measuring $g$ with a simple pendulum, L-length, T-oscillation period

\[ T = 2\pi \left( \frac{L}{g} \right)^{1/2} \quad \Rightarrow \quad g = 4 \pi^2 \frac{L}{T^2} \]

\[ \frac{\delta (T^2)}{T^2} = 2 \frac{\delta T}{T} \quad \Rightarrow \quad \frac{\delta g}{g} = \sqrt{\left( \frac{\delta L}{L} \right)^2 + \left( 2 \frac{\delta T}{T} \right)^2} \]

$L = 92.95 \pm 0.1 \text{ cm}$, $T = 1.936 \pm 0.004 \text{ sec.}$

\[ g_{\text{best}} = \frac{4 \pi^2 \times (92.95 \text{ cm})}{(1.936 \text{ sec})^2} = 979 \text{ cm/sec}^2 \]

\[ \frac{\delta L}{L} = 0.1\% \quad \frac{\delta T}{T} = 0.2\% \quad \frac{\delta g}{g} = \sqrt{(0.1)^2 + (2 \times 0.2)^2} \% = 0.4\% \]

\[ \delta g = 0.004 \times 979 \text{ cm/sec}^2 = 4 \text{ cm/sec}^2 \]
Systematic and Random Errors

Random: small
Systematic: small

Random: large
Systematic: small

Random: small
Systematic: large

Random: large
Systematic: large
Systematic and Random Errors

Real experiment
Statistical Analysis of Random Errors

Example: 5 measurements of some value.

\[ 71, 72, 72, 73, 71 \]

\[ \bar{x} = \frac{71 + 72 + 72 + 73 + 71}{5} = 71.8 = \frac{\sum_{i=1}^{N} X_j}{N} \]

<table>
<thead>
<tr>
<th>Trial number</th>
<th>Measured value</th>
<th>Deviation ( d_i )</th>
<th>Deviation squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>71</td>
<td>-0.8</td>
<td>0.64</td>
</tr>
<tr>
<td>2</td>
<td>72</td>
<td>0.2</td>
<td>0.04</td>
</tr>
<tr>
<td>3</td>
<td>72</td>
<td>0.2</td>
<td>0.04</td>
</tr>
<tr>
<td>4</td>
<td>73</td>
<td>1.2</td>
<td>1.22</td>
</tr>
<tr>
<td>5</td>
<td>71</td>
<td>-0.8</td>
<td>0.64</td>
</tr>
</tbody>
</table>

\[ d_i = x_i - \bar{x} \]

\[ \sum d_i = 0 \]

\[ \sum d_i^2 = 2.8 \]
Statistical Analysis of Random Errors

Linear deviations are no good to characterize the statistics of random errors: have zero average, let’s go for squares.

Definition: Standard deviation $\sigma_x$:

$$
\sigma_x = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (d_i)^2} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2}
$$

Root mean square (RMS) deviation of the measurements $x_1, x_2, \ldots, x_N$
Experimental errors should be always rounded to one significant digit.

\[ g = 9.82 \pm 0.02385 \quad \text{wrong} \]

\[ g = 9.82 \pm 0.02 \quad \text{correct} \]

Thus error calculations become simple estimates.

*Exception*: if the leading significant digit of the error is 1, keep one digit more.

Everest is \( 8848 \pm 1.5 \text{ m} \) high.
The last significant figure in the answer should be the same order of magnitude as the uncertainty.

\[ 92.8 \pm 0.3 \]

\[ 92 \pm 3 \]

\[ 90 \pm 30 \]

During the calculation retain one more digit than is finally justified to decrease the error.
Summary

- **Systematic error**: Reproducible inaccuracy introduced by faulty equipment, calibration or technique.
- **Random error**: Indefiniteness of result introduced by finite precision of measurement or statistical variations. Measure of fluctuation after repeated experimentation.
- **Uncertainty**: Magnitude of error that is estimated to have been made in determination of results.
- **Accuracy**: Measure of how close the result of an experiment comes to the “true” value.
- **Precision**: Measure of how carefully the result is determined without reference to any “true” value.
# Summary - propagation of errors

General formula for propagation of error: If \( q = q(x_i, x_{i+1}, \ldots, x_n) \) is any function of \( x_i, x_{i+1}, \ldots, x_n \), then

\[
\delta q = \sqrt{\left( \frac{\partial q}{\partial x_i} \delta x_i \right)^2 + \cdots + \left( \frac{\partial q}{\partial x_n} \delta x_n \right)^2}
\]

(Provided all errors are independent and random)

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative(s)</th>
<th>Variance</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f = kx ); ( k \in \mathbb{R} )</td>
<td>( \frac{\partial f}{\partial x} = k )</td>
<td>( \sigma_f^2 = k^2 \sigma_x^2 )</td>
<td>( \sigma_f = k \sigma_x )</td>
</tr>
<tr>
<td>( f = x + y )</td>
<td>( \frac{\partial f}{\partial x} = 1 ) and ( \frac{\partial f}{\partial y} = 1 )</td>
<td>( \sigma_f^2 = \sigma_x^2 + \sigma_y^2 )</td>
<td>( \sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2} )</td>
</tr>
<tr>
<td>( f = x - y )</td>
<td>( \frac{\partial f}{\partial x} = 1 ) and ( \frac{\partial f}{\partial y} = -1 )</td>
<td>( \sigma_f^2 = \sigma_x^2 + \sigma_y^2 )</td>
<td>( \sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2} )</td>
</tr>
<tr>
<td>( f = xy )</td>
<td>( \frac{\partial f}{\partial x} = y ) and ( \frac{\partial f}{\partial y} = x )</td>
<td>( \left( \frac{\sigma_f}{f} \right)^2 = \left( \frac{\sigma_x}{x} \right)^2 + \left( \frac{\sigma_y}{y} \right)^2 )</td>
<td>( \sigma_f = f \sqrt{\left( \frac{\sigma_x}{x} \right)^2 + \left( \frac{\sigma_y}{y} \right)^2} )</td>
</tr>
<tr>
<td>( f = x / y )</td>
<td>( \frac{\partial f}{\partial x} = \frac{1}{y} ) and ( \frac{\partial f}{\partial y} = -\frac{x}{y^2} )</td>
<td>( \left( \frac{\sigma_f}{f} \right)^2 = \left( \frac{\sigma_x}{x} \right)^2 + \left( \frac{\sigma_y}{y} \right)^2 )</td>
<td>( \sigma_f = f \sqrt{\left( \frac{\sigma_x}{x} \right)^2 + \left( \frac{\sigma_y}{y} \right)^2} )</td>
</tr>
</tbody>
</table>
Summary-statistical error

Statistical Error: If $x_i, x_{i+1}, \ldots, x_n$ are $N$ measurements of the same quantity $x$, all using the same method. If all the uncertainties are random and small,

The best estimate of $x$ (mean)

$$
\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i
$$

The average uncertainty of individual $x_i, x_{i+1}, \ldots, x_n$ is given by standard deviation, or SD:

$$
\sigma_x = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2}
$$
Estimate and Try to keep error small

Source: http://antongerdelan.net/teaching/vis/datareps
Qualitative Error in Slope

\[ \Delta S = \frac{1}{2} |S_1 - S_2| \]
Suppose you have measurements of \((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\) and you want to fit it with the best straight line \(y = Bx + A\).

For this one has to minimize the \(\chi^2\) defined as:

\[
\chi^2 = \sum_{i=1}^{N} \frac{(y_i - A - Bx_i)^2}{\sigma_y^2}
\]

And the partial derivatives are:

\[
\frac{\partial \chi^2}{\partial A} = \frac{-2}{\sigma_y^2} \sum_{i=1}^{N} (y_i - A - Bx_i) = 0
\]

\[
\frac{\partial \chi^2}{\partial B} = \frac{-2}{\sigma_y^2} \sum_{i=1}^{N} x_i (y_i - A - Bx_i) = 0
\]

\[
A = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta}
\]

\[
B = \frac{N \sum xy - \sum x \sum y}{\Delta}
\]

\[
\Delta = N \sum x^2 - (\sum x)^2.
\]
An example of best line fit

Suppose a spring is hanging vertically and you put different mass on the bottom of the spring. The spring will extend to different lengths. Here mass is like “x” and the length the spring extends is “y”.

Table 8.1. Masses $m_i$ (in kg) and lengths $l_i$ (in cm) for a spring balance. The “x” and “y” in quotes indicate which variables play the roles of x and y in this example.

<table>
<thead>
<tr>
<th>Trial number $i$</th>
<th>“x” Load, $m_i$</th>
<th>“y” Length, $l_i$</th>
<th>$m_i^2$</th>
<th>$m_i l_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>42.0</td>
<td>4</td>
<td>84</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>48.4</td>
<td>16</td>
<td>194</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>51.3</td>
<td>36</td>
<td>308</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>56.3</td>
<td>64</td>
<td>450</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>58.6</td>
<td>100</td>
<td>586</td>
</tr>
<tr>
<td>$N = 5$</td>
<td>$\Sigma m_i = 30$</td>
<td>$\Sigma l_i = 256.6$</td>
<td>$\Sigma m_i^2 = 220$</td>
<td>$\Sigma m_i l_i = 1,622$</td>
</tr>
</tbody>
</table>

\[ \Delta = N\Sigma x^2 - (\Sigma x)^2. \]

\[ \Delta = N \Sigma m^2 - (\Sigma m)^2 \]
\[ = 5 \times 220 - 30^2 = 200. \]
An example of best line fit (cont.)

\[ A = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta} \]

\[ B = \frac{N \sum xy - \sum x \sum y}{\Delta} \]

\[ A = \frac{\sum m^2 \sum l - \sum m \sum ml}{\Delta} \]

\[ = \frac{220 \times 256.6 - 30 \times 1622}{200} \approx 39.0 \text{ cm.} \]

\[ B = \frac{N \sum ml - \sum m \sum l}{\Delta} \]

\[ = \frac{5 \times 1622 - 30 \times 256.6}{200} \approx 2.06 \text{ cm/kg.} \]

![Graph showing a linear relationship between Mass and Length, with intercept A and slope B.](image)
Semester & next two weeks plan

• There are nine experiments total in this semester.
• The lab manuals are on Physics Dept website
  http://physics.niser.ac.in/act.php (under lab manual in semester-I (P-141)
• Read the lab manual before coming to class
• There will be lab teacher, scientists, PhD students and lab operator to help you during lab class.

Next two weeks Plan:

• You will do two experiments in next two weeks. Everybody will do the same two experiments to know how to use error analysis properly. You will submit the report of these two experiments also and it will be graded
• In the first week first 50 students will do expt-I and the other 50 will do expt-II.
• In the second week, it will be swapped.
Expt.1: Error Analysis Training

- ‘g’ by free fall
- You measure the time ‘T’ of free fall large number of times $T_1, T_2, \ldots, T_N$
- Calculate corresponding the ‘$g_1, g_2, \ldots, g_N$’
- Find the mean ‘g’ and standard deviation in ‘g’
- Report the ‘g’ value with measurement error
- Is there any systematic error?
Expt.2: Error Analysis Training

• Volume of the metal bar
• Determine the least count of vernier calipers, screw gauze and travelling microscope
• Measure length, breadth and thickness of metal bar using vernier calipers and screw gauze
• Calculate volume and calculate error in volume by using error propagation formula
• Use travelling microscope to find the volume of metal bar by measuring the volume change of water
Reference

*An Introduction to Error Analysis:*
John R. Taylor

Additional References

*A Practical Guide to Data Analysis for Physical Science Students:* Louis Lyons

*Data Reduction and Error Analysis:*
P. R. Bevington and D. K. Robinson

*Practical Physics:* G L Squires
List of Experiments

• ‘g’ by compound pendulum
• Young’s modulus by bending of a beam
• Specific heat of Graphite
• Thermal conductivity of a poor conductor
• Viscosity of liquids by falling ball method
• Surface tension by capillary rise method
• Determination of ‘J’, Joule constant by electrical method
• Standing waves and spring constant of a soft massive spring
• Moment of Inertia of different bodies
Lab Evaluation P141

(1) Weekly Lab work and report submission (50 marks)
   -> Aim of the experiment (one or two lines)
   -> Theoretical introduction, what formula used, apparatus, diagrams
   -> Data (table)
   -> Analysis (plots, error calculations)
   -> Summary/Conclusion

(2) Viva (20 marks)

(3) Final Examination (30 marks)
   You will be given one of the experiments which you performed in the laboratory during semester, in the final exam
(1) Weekly Lab work and report submission (50)

-> If you perform an experiment this week, you need to submit the report the following week (at the beginning of lab class). So, you have one week to prepare your lab report.

-> If you miss to submit the report in the following lab, 50% of the lab report mark will be taken off.

-> If you submit the report after 2 weeks, 75% will be taken off.

-> After 2 weeks, Lab reports will not be accepted.

(in case you are sick or you have any other reason that stops you from submitting report or performing lab, send email to your TA’s/ Teacher before lab class, not after)