On the Fourier Expansions of Jacobi Forms of Half-Integral Weight

Brundaban Sahu

(joint work with B. Ramakrishnan)

Harish-Chandra Research Institute,
Allahabad

Notations and Definitions

Let k, N be a positive integers
χ be a Dirichlet character modulo $4N$.
Let $\mathcal{H}$ denote the complex upper half-plane.
For a complex number z,

$$\sqrt{z} = |z|^{1/2} e^{\frac{i}{2} \text{arg } z}, \text{ with } -\pi < \text{arg } z \leq \pi.$$

$$z^{k/2} = (\sqrt{z})^k \quad \text{for any} \quad k \in \mathbf{Z}.$$

For $z \in \mathbf{C}$ and $a, b \in \mathbf{Z}$, we put $e_b^a(z) = e^{2\pi i a z / b}$.

A *Jacobi form* $\phi(\tau, z)$ of weight $k+1/2$ and index m for the group $\Gamma_0(4N)$, with character χ, is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following conditions. (i)

$$\phi\Big|_{k,m}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) = \chi(d)\phi(\tau, z),$$

where

$$\phi\Big|_{k,m}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) = \left(\frac{c}{d}\right)\epsilon_d^{2k+1}(c\tau+d)^{-k-1/2}$$

$$\cdot\ e^m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)$$

$$\cdot\ \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ and $\lambda, \mu \in \mathbb{Z}$, $\left(\frac{c}{d}\right)$ denotes the Jacobi symbol and $\epsilon_d = 1$ *or* i according as d is 1 or 3 modulo 4.

(ii) For every $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$(c\tau + d)^{-k-1/2} e^m \left(\frac{-cz^2}{c\tau + d}\right) \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$$

has a Fourier development of the form

$$\sum_{\substack{n,r \in \mathbb{Q} \\ r^2 \leq 4nm}} c_{\phi,\alpha}(n,r)\ e\,(n\tau + rz)\,,$$

where the sum varies over rational numbers n, r with bounded denominators subject to the condition $r^2 \leq 4nm$.

Further, if $c_{\phi,\alpha}(n,r)$ satisfies the condition $c_{\phi,\alpha}(n,r) \neq 0$ implies $r^2 < 4nm$, then ϕ is called a *Jacobi cusp form*.

We denote by $J_{k+1/2,m}(4N, \chi)$, the space of Jacobi forms of weight $k + 1/2$, index m for $\Gamma_0(4N)$ with character χ.

Let $\phi(\tau, z)$ be a Jacobi form in $J_{k+1/2,m}(4N, \chi)$. Then its Fourier expansion at the infinite cusp is of the form

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4nm}} c_\phi(n, r) e(n\tau + rz). \qquad (1)$$

We have

Lemma 1. Let $\phi \in J_{k+1/2,m}(4N, \chi)$ having a Fourier expansion as in (1). Then $c_\phi(n, r)$ depends only on $4mn - r^2$ and on $r \pmod{2m}$, i.e $c_\phi(n, r) = c_\phi(n', r')$ whenever $r \equiv r'$ $(2m)$ and $4mn - r^2 = 4mn' - r'^2$

The Lemma follows using similar arguments as in the case of integral weight. (See for instance Eichler-Zagier, Prog. in Math. **55**.)

For $D \leq 0$, $\mu \pmod{2m}$, define $c_\mu(|D|)$ as follows:

$$c_\mu(|D|) := \begin{cases} c_\phi(\frac{r^2-D}{4m}, r) & \text{if } D \equiv \mu^2 \pmod{4m}, \\ & \quad r \equiv \mu \pmod{2m}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

For $\mu \pmod{2m}$, let

$$h_\mu(\tau) := \sum_{|D|=0}^{\infty} c_\mu(|D|) e_{4m}(|D|\tau) . \tag{3}$$

For $\alpha \pmod{2m}$, the Jacobi theta function is given by

$$\vartheta_\alpha(\tau, z) = \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \alpha \pmod{2m}}} e\left(\frac{r^2}{4m}\tau + rz\right) . \tag{4}$$

By standard arguments (as in the case of integral weight), the Jacobi form $\phi \in J_{k+1/2,m}(4N,\chi)$ can be expressed as follows.

$$\phi(\tau, z) = \sum_{\mu \pmod{2m}} h_\mu(\tau)\vartheta_\mu(\tau, z), \qquad (5)$$

where $(h_\mu)_{\mu \pmod{2m}}$ is a vector valued modular form of weight k.

Some Transformation Properties of $h_\mu(\tau)$

Lemma 2. (Y. Tanigawa) Let $\phi(\tau, z) \in J_{k+1/2,m}(4N,\chi)$ and let h_α, $\alpha \pmod{2m}$ be a component of ϕ. Then the following formulas hold for $h_\alpha(\tau)$.

$$(i) \qquad h_{-\alpha}(\tau) = \chi(-1)h_\alpha(\tau).$$

$$(ii) \qquad h_\alpha(\tau+b)e_{4m}(\alpha^2 b) = h_\alpha(\tau), \qquad \textit{for any } b \in \mathbb{Z}.$$

$$(iii) \quad (4N\tau+1)^k\, h_\alpha(\tau) = \sum_{\beta \pmod{2m}} \xi_{\alpha,\beta}\, h_\beta\left(\frac{\tau}{4N\tau+1}\right)$$

or equivalently

$$(-4N\tau+1)^{-k}\ h_\alpha\left(\frac{\tau}{-4N\tau+1}\right)=\sum_{\beta \pmod{2m}} \xi_{\alpha,\beta}\, h_\beta(\tau),$$

where

$$\xi_{\alpha,\beta}=\frac{1}{2m}\sum_{\gamma \pmod{2m}} e_{4m}(-4N\gamma^2+2\gamma(\beta-\alpha)). \tag{6}$$

Remark 1. The above lemma gives the relationship among the $h_\alpha(\tau)$. If one of the $h_\alpha(\tau)$ is zero, then there is a linear dependence equation among the $h_\alpha(\tau)$ which is given by the transformation rule (iii) of Lemma 2. It is natural to ask about the maximum number of components $h_\alpha(\tau)$ that can be zero in order that the given form ϕ is non-zero. Our theorems are motivated by this question considered by Skogman for the Jacobi forms of integral weight. Note that if χ is an odd character, i.e., $\chi(-1) = -1$ then $h_0(\tau) = 0 = h_m(\tau)$, which follows from (i) of the above lemma.

Remark 2. Let $\phi(\tau, z) \in J_{k+1/2,1}(4N, \chi)$ be a Jacobi form of index 1. Using (i) of above Lemma, we see that when χ is an odd character, then $\phi(\tau, z) = 0$. Therefore, there is no non-zero Jacobi form of weight $k + 1/2$ and index 1 when the character χ is odd.

Remark 3. Let $m = 2$. In this case the Jacobi form $\phi(\tau, z) \in J_{k+1/2,2}(4N, \chi)$ will have four components $h_\mu(\tau)$, $\mu = 0, 1, 2, 3$. When χ is an odd character, then using (i) of above Lemma we get $h_0 = 0 = h_2$ and $h_1 = -h_3$. Therefore, $h_1(\tau)$ or $h_3(\tau)$ determines $\phi(\tau, z) \in J_{k+1/2,2}(4N, \chi)$ when χ is odd.

Now, let $\phi(\tau, z) \in J_{k+1/2,2}(4N, \chi)$, N odd and χ be an even character. In this case, $h_1 = h_3$. Also, it can be seen that $\xi_{0,1} = \xi_{0,3} = 0$ and $\xi_{0,2} = 1$. Using this in the transformation (iii) of above Lemma and substituting $\alpha = 0$ and assuming that $h_0 = 0$, we get

$$\begin{aligned} 0 &= \xi_{0,1} h_1(\tau) + \xi_{0,2} h_2(\tau) + \xi_{0,3} h_3(\tau) \\ &= 2\xi_{0,1} h_1(\tau) + \xi_{0,2} h_2(\tau) \\ &= h_2(\tau). \end{aligned}$$

Therefore, $h_0 = 0$ implies that $h_2 = 0$. Already we have seen that h_1 determines h_3. This shows that $h_0(\tau)$ and $h_1(\tau)$ determine $\phi(\tau, z)$ when $m = 2$, χ is even and N is odd.

We prove the following theorems regarding the number of components h_μ that determine a Jacobi form.

Theorem 1. Let $\phi(\tau, z) \in J_{k+1/2,p}(4N, \chi)$, where p is an odd prime such that $\gcd(N, 2p) = 1$. For some $\alpha, \beta \pmod{2p}$ with $2 \nmid \alpha$, $2|\beta$ and $\gcd(\alpha\beta, p) = 1$, if we have $h_\alpha(\tau) = 0$ and $h_\beta(\tau) = 0$, then $\phi(\tau, z) = 0$.

Theorem 2. Let $\phi(\tau, z) \in J_{k+1/2,p}(4N, \chi)$, where p is an odd prime such that $p|N$. Then among the $2p$ components $h_\mu(\tau)$, λ_χ of them determine the Jacobi form $\phi(\tau, z)$, where

$$\lambda_\chi = \begin{cases} p-1 & \text{if } \chi \text{ is odd,} \\ p+1 & \text{if } \chi \text{ is even.} \end{cases}$$

Proof of Theorem 1.

Lemma 3. Let m be an odd natural number such that $\gcd(m,N) = 1$ and let $\xi_{\alpha,\beta}$ be the Gauss sum as defined in (6) Then

$$\xi_{\alpha,\beta} = \begin{cases} \frac{1}{\sqrt{m}}\left(\frac{-N}{m}\right)\epsilon_m \; e_{4m}\left((4N)^{-1}(\beta-\alpha)^2\right) \\ \qquad\qquad\qquad \text{if } 2|(\beta-\alpha), \\ 0 \qquad\qquad \text{otherwise,} \end{cases} \tag{7}$$

where $(4N)^{-1}$ is an integer which is the inverse of $4N$ modulo m and $\epsilon_m = 1$ or i according as $m \equiv 1$ or 3 (mod 4).

We observe that

$$\begin{aligned} \xi_{\alpha,-\beta} &= \xi_{-\alpha,\beta}, \\ \xi_{\alpha,\beta} &= \xi_{-\alpha,-\beta}. \end{aligned} \tag{8}$$

Assume that for an odd α, $0 \le \alpha \le 2p$ with $\gcd(\alpha, p) = 1$ we have $h_\alpha(\tau) = 0$. We will show that $h_\mu(\tau) = 0$ for all odd $\mu \pmod{2p}$. By Lemma 2, (i) we have $h_{-\alpha}(\tau) = 0$. Substituting $h_\alpha = 0 = h_{-\alpha}$ in the equivalent form of Lemma.2 (iii), we get

$$\begin{aligned} \sum_\beta \xi_{\alpha,\beta} h_\beta(\tau) = 0, \\ \sum_\beta \xi_{-\alpha,\beta} h_\beta(\tau) = 0. \end{aligned} \tag{9}$$

Observe that in the expansion of each of the $h_\beta(\tau)$, only $h_\beta(\tau)$ and $h_{-\beta}(\tau)$ have terms of the form $e\left(\frac{-\beta^2}{4p}\tau\right) e(n\tau)$ (since $x \equiv \mu^2 \pmod{4p}$ has only two solutions $\pm\mu$ modulo $2p$, if $\gcd(\mu, p) = 1$). Using this in the above inversion formulas, we get

$$\begin{aligned} \xi_{\alpha,\beta} h_\beta(\tau) + \xi_{\alpha,-\beta} h_{-\beta}(\tau) = 0, \\ \xi_{-\alpha,\beta} h_\beta(\tau) + \xi_{-\alpha,-\beta} h_{-\beta}(\tau) = 0. \end{aligned} \tag{10}$$

Hence,

$$h_\beta(\tau) = -\frac{\xi_{\alpha,-\beta}}{\xi_{\alpha,\beta}} h_{-\beta}(\tau) = -\frac{\xi_{-\alpha,-\beta}}{\xi_{-\alpha,\beta}} h_{-\beta}(\tau). \tag{11}$$

Claim 1. For $\beta \pmod{2p}$ with $\gcd(\beta, 2p) = 1$, we have

$$\xi_{-\alpha,\beta}\xi_{\alpha,-\beta} \neq \xi_{\alpha,\beta}\xi_{-\alpha,-\beta} \quad \text{for } \alpha \equiv \beta \pmod 2.$$

From this it follows that $h_{-\beta}(\tau) = 0 = h_\beta(\tau)$ when $\gcd(\beta, 2p) = 1$.

Proof of Claim 1. Assume the contrary, i.e., assume that

$$\xi_{-\alpha,\beta}\xi_{\alpha,-\beta} = \xi_{\alpha,\beta}\xi_{-\alpha,-\beta}.$$

Then it follows that

$$\xi_{\alpha,\beta}^2 = \xi_{\alpha,-\beta}^2,$$

from which we should have

$$\xi_{\alpha,\beta} = \pm\xi_{\alpha,-\beta}. \tag{12}$$

We shall now show that the last identity is not true when $\gcd(\beta, 2p) = 1$.

By Lemma 3 we get

$$\xi_{\alpha,-\beta} = \frac{1}{\sqrt{p}}\left(\frac{-N}{p}\right)\epsilon_p\ e_{4p}\left(N'(\alpha+\beta)^2\right),$$

if $2|(\alpha+\beta)$, where N' is an integer such that $(4N)N' \equiv 1 \pmod{p}$. Therefore, for (12) to be true we must have

$$e_p(-N'\alpha\beta) = \pm 1,$$

which implies that $p|N'\alpha\beta$, a contradiction. Similarly, if one starts with $h_\alpha = 0$ for even α, then $h_\beta = 0$ for all β even with $\gcd(\beta, p) = 1$. It remains to show that $h_0(\tau) = 0 = h_p(\tau)$. If χ is an odd character then as remarked in Remark.1, we have $h_0(\tau) = 0 = h_p(\tau)$.

So, let χ be an even character. If $h_\alpha(\tau) = 0$ for an even α, then applying the inversion formula as before and observing that $x^2 \equiv 0 \pmod{4p}$ has only one solution $x = 0$ modulo $2p$, we have $\xi_{\alpha,0} h_0(\tau) = 0$. Since $\xi_{\alpha,0} \neq 0$, it follows that $h_0(\tau) = 0$. Similarly, one can prove that $h_p(\tau) = 0$ by considering the square class p^2 modulo $4p$ which has only one solution $(= p)$ modulo $2p$. This completes the proof of Theorem 1.

Proof of Theorem 2.

Here the index is p and $p|N$. In this case we have

$$\begin{aligned}\xi_{\alpha,\beta} &= \frac{1}{2p}\sum_{\gamma \pmod{2p}} e_{2p}\Big(\gamma(\beta-\alpha)\Big) \\ &= \begin{cases} 1 & \text{if } 2p|(\beta-\alpha), \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{13}$$

In other words, $\xi_{\alpha,\beta} \neq 0$ if and only if $\alpha \equiv \beta \pmod{2p}$. Therefore property (iii) of Lemma 2. won't give any dependency relation among the $h_\alpha(\tau)$. Now let χ be an odd character. In this case, as observed before we have $h_0(\tau) = 0 = h_p(\tau)$. Also, by the same property (i) of Lemma 2, we have $h_\alpha(\tau) = -h_{-\alpha}(\tau)$. So, in the remaining $2p-2$ components, if we assume that $p-1$ of them are zero, then $\phi(\tau,z) = 0$. This completes the first part. If χ is an even character, we cannot conclude that h_0 and h_p are zero. Therefore, combining with the property $h_\alpha(\tau) = h_{-\alpha}(\tau)$ we see that one needs $p-1+2 = p+1$ components h_α to determine the given Jacobi form ϕ.

Theorem 3. Let $\phi(\tau, z) \in J_{k+1/2,pq}(4N, \chi)$, where p, q are distinct odd primes. Then the components h_α and h_β with $2|\alpha$, $2 \not| \beta$ and $\gcd(\alpha\beta, pq) = 1$ determine the associated Jacobi form $\phi(\tau, z)$.

Theorem 4. Let $\phi(\tau, z) \in J_{k+1/2,p^2}(4N, \chi)$, where p is an odd prime such that $\gcd(p, N) = 1$. Then among the $2p^2$ components $h_\mu(\tau)$, the following components

$$\{h_a, h_b\}, 2|a, 2 \not| b, \ \gcd(ab, p) = 1 \text{ and}$$

$$\left\{h_{2ip}, h_{(2i+1)p}\right\}_{i=0}^{(p-1)/2},$$

when χ is even and

$$\{h_a, h_b\}, 2|a, 2 \not| b, \ \gcd(ab, p) = 1 \text{ and}$$

$$\left\{h_{2(i+1)p}\right\}_{i=0}^{(p-3)/2}, \left\{h_{(2i-1)p}\right\}_{i=1}^{(p-1)/2},$$

when χ is odd determine the Jacobi form $\phi(\tau, z)$. In other words, at most $p+1$ or $p+3$ (according as χ is odd or even) of the components h_μ will determine the Jacobi form $\phi(\tau, z)$.

Reference:
On the Fourier Expansions of Jacobi Forms of Half-Integral Weight,
B. Ramakrishnan and Brundaban Sahu
Int. J. Math. Math. Sci. Vol 2006

THANK YOU