

EXPLICIT EVALUATION OF TRIPLE CONVOLUTION SUMS OF THE DIVISOR FUNCTIONS

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ABSTRACT. In this paper, we use the theory of modular forms and give a general method to obtain the convolution sums

$$W_{d_1, d_2, d_3}^{r_1, r_2, r_3}(n) = \sum_{\substack{l_1, l_2, l_3 \in \mathbb{N} \\ d_1 l_1 + d_2 l_2 + d_3 l_3 = n}} \sigma_{r_1}(l_1) \sigma_{r_2}(l_2) \sigma_{r_3}(l_3),$$

for odd integers $r_1, r_2, r_3 \geq 1$, and $d_1, d_2, d_3, n \in \mathbb{N}$, where $\sigma_r(n)$ is the sum of the r -th powers of the positive divisors of n . We consider four cases, namely (i) $r_1 = r_2 = r_3 = 1$, (ii) $r_1 = r_2 = 1; r_3 \geq 3$ (iii) $r_1 = 1; r_2, r_3 \geq 3$ and (iv) $r_1, r_2, r_3 \geq 3$, and give explicit expressions for the respective convolution sums. We provide several examples of these convolution sums in each case and further use these formulas to obtain explicit formulas for the number of representations of a positive integer n by certain positive definite quadratic forms. The existing formulas for $W_{1,1,1}(n)$ (in [20]), $W_{1,1,2}(n)$, $W_{1,2,2}(n)$, $W_{1,2,4}(n)$ (in [7]), $W_{1,1,1}^{1,3,3}(n)$, $W_{1,1,3}^{1,3,3}(n)$, $W_{1,3,3}^{1,3,3}(n)$, $W_{3,1,1}^{1,3,3}(n)$, $W_{3,3,1}^{1,3,3}(n)$ (in [35]), $W_{d_1, d_2, d_3}(n)$, $\text{lcm}(d_1, d_2, d_3) \leq 6$ (in [30]) and $\text{lcm}(d_1, d_2, d_3) = 7, 8, 9$ (in [31]), which were all obtained by using the theory of quasimodular forms, follow from our method.

1. INTRODUCTION

Let \mathbb{N} denote the set of natural numbers and for $s, n \in \mathbb{N}$, define

$$\sigma_s(n) := \sum_{d|n} d^s,$$

where $d|n$ runs through the positive integers. We set $\sigma_s(n) = 0$ when $n \notin \mathbb{N}$ and also write $\sigma(n)$ for $\sigma_1(n)$. Let $t \geq 2$ be a natural number and $n, d_i, k_i \in \mathbb{N}, 1 \leq i \leq t$ be such that k_i 's are even, $\gcd(d_1, d_2, \dots, d_t) = 1$. Then the convolution sum $W_{d_1, d_2, \dots, d_t}^{k_1-1, k_2-1, \dots, k_t-1}(n)$ of the divisor functions, is defined by

$$W_{d_1, d_2, \dots, d_t}^{k_1-1, k_2-1, \dots, k_t-1}(n) = \sum_{\substack{l_1, l_2, \dots, l_t \in \mathbb{N} \\ d_1 l_1 + d_2 l_2 + \dots + d_t l_t = n}} \sigma_{k_1-1}(l_1) \sigma_{k_2-1}(l_2) \cdots \sigma_{k_t-1}(l_t). \quad (1)$$

These types of sums have been studied since the mid-nineteenth century and the explicit expressions for these convolution sums have been obtained using various methods namely, elementary evaluation, using the theory of modular forms and quasimodular forms, and

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also using (p, k) parametrization, etc. We refer to the book by K. S. Williams [45] for more details about the history of this problem. For $t = 3$, we shall refer to these convolution sums as the triple convolution sums. The purpose of this paper is to give a general method of explicit evaluation of these convolution sums using the theory of modular forms. In our earlier work [38] we considered an extension of the Ramanujan-Serre derivative map to derive explicit evaluation of the above type of convolution sums when $t = 2$ (involving two divisor functions) using the theory of modular forms. Our evaluation of triple convolution sums ($t = 3$) derived in this work involve the convolution sums with two divisor functions ($t = 2$), i.e., convolution sums of the type $W_{d_1, d_2}^{k_1-1, k_2-1}(n)$, for even integers $k_1, k_2 \geq 2$ and $d_1, d_2 \in \mathbb{N}$, with $\gcd(d_1, d_2) = 1$. Therefore, we shall first give a brief history of the case $t = 2$ and then describe our work in this paper for the case $t = 3$.

1.1. The case when $t = 2$ and $k_1 = k_2 = 2$. In this case, we simply denote the sum by $W_{d_1, d_2}(n)$ instead of $W_{d_1, d_2}^{1,1}(n)$. The convolution sum $W_{d_1, d_2}(n)$ has been evaluated by elementary methods for smaller values of d_1, d_2 and it has been extensively evaluated by using the theory of quasimodular forms for large values of d_1, d_2 . The latter method uses the fact that $W_{d_1, d_2}(n)$ is a part of the Fourier coefficient of certain quasimodular form of level $N = d_1 d_2$ and weight 4. We give below a list of references for the explicit computation of these sums for the values $d_1 = 1, 1 \leq d_2 \leq 29$ ($d_2 \neq 19, 26$), $d_2 = 32, 36, 41, 47, 59, 71$ and for the pairs $(d_1, d_2) \in \{(2, 3), (2, 5), (2, 7), (2, 9), (2, 11), (3, 4), (3, 5), (3, 7), (3, 8), (4, 5), (4, 7), (4, 9)\}$ (cf. [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 21, 26, 27, 28, 29, 32, 33, 36, 39, 40, 42, 43, 44]).

Remark 1.1. In our recent work [38], we demonstrated a general procedure for the evaluation of the sums $W_{d_1, d_2}(n)$ using the theory of modular forms, which is given in (11). As examples, sums corresponding to the cases $(d_1, d_2) \in \{(1, 30), (2, 15), (3, 10), (5, 6)\}$ were given explicitly.

1.2. The case when $t = 2$, $k_1 = 2$ and $k_2 = k \geq 4$ (even). The convolution sums $W_{d_1, d_2}^{1, k-1}(n)$ have been computed by several mathematicians mostly using the theory of quasimodular forms. Like in the previous case, we give below a list of references for the explicit computation of the sum $W_{d_1, d_2}^{1, k-1}(n)$ for the values $(d_1, d_2) \in \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (1, 6), (6, 1), (2, 3), (3, 2)\}$, when $k = 4$; $(d_1, d_2) \in \{(1, 1), (1, 2), (2, 1), (1, 4), (4, 1)\}$, when $k = 6, 8, 10$; and $(d_1, d_2) = (1, 1)$, when $k = 12$ (cf. [12, 17, 18, 19, 20, 24, 25, 39, 40, 46]).

Remark 1.2. Apart from the works mentioned as above, in our recent work [38], we derived a general formula for this type of convolution sums. Explicit expression for this sum is given in (10). Using our method, we obtained some new examples for the sums $W_{d_1, d_2}^{1, k-1}(n)$, corresponding to $(d_1, d_2) \in \{(1, 3), (3, 1), (1, 6), (6, 1), (2, 3), (3, 2)\}$ and $k = 4, 6, 8, 10$. We refer to [38, Remark 4.4] concerning the earlier works related to these cases when $k = 4$.

We now present briefly the method used in our earlier work [38] and then describe the extension of work considered in the present paper. To derive the above type of convolution sums, we used a simple extension of the Ramanujan-Serre derivative map. Ramanujan showed that $DE_2(z) - \frac{1}{12}E_2^2(z)$ is a modular form of weight 4 (and hence this is equal to $\pm\frac{1}{12}E_4(z)$), where $D = \frac{1}{2\pi i}\frac{d}{dz}$. When f is a modular form of weight k for the full modular group, an extension of the above map is given by

$$\vartheta_k(f)(z) = Df(z) - \frac{k}{12}E_2(z)f(z), \quad (2)$$

which is a modular form of weight $k+2$ for the full modular group, and this is referred to as the *Ramanujan-Serre derivative map*. If f is a cusp form, then $\vartheta_k(f)$ is also a cusp form. (This result also holds when f is a modular/cusp form with level and character.) In our recent work [38], we gave a simple extension of the Ramanujan-Serre derivative map ϑ_k and also derived a generalisation of Ramanujan's result mentioned above. More precisely, when d_1, d_2 are positive integers such that $\gcd(d_1, d_2) = 1$, we proved that the function

$$\frac{6d_1}{d_2}DE_2(d_1z) + \frac{6d_2}{d_1}DE_2(d_2z) - E_2(d_1z)E_2(d_2z) \quad (3)$$

is a modular form in the space $M_4(d_1d_2)$. As an application, we derived the convolution sum $W_{d_1, d_2}(n)$ by using a basis for the space of modular forms $M_4(d_1d_2)$. Using this identity, we further evaluated convolution sums of the type $\sum_{\substack{l, m \in \mathbb{N}, \\ d_1l + d_2m = n}} l\sigma(l)\sigma(m)$. We also

proved that for positive integers d_1, d_2 , the differential operator $\vartheta_{k;(d_1, d_2)}$ defined by

$$\vartheta_{k;(d_1, d_2)}(f) := d_2Df(d_2z) - \frac{k}{12}d_1E_2(d_1z)f(d_2z) \quad (4)$$

maps the space $M_k(M, \chi)$ into the space $M_{k+2}(N, \chi)$, where $N = \text{lcm}(Md_2, d_1)$ and χ is a Dirichlet character modulo N . In particular, by taking $f(z) = E_k(z)$, the Eisenstein series of weight k for the full modular group, we derived the convolution sums $W_{d_1, d_2}^{1, k-1}(n)$, using a basis for the space $M_{k+2}(N)$. For even $k \geq 4$, we further used this operator to evaluate the convolution sum $\sum_{\substack{l, m \in \mathbb{N}, \\ d_1l + d_2m = n}} l\sigma(l)\sigma_{k-1}(m)$.

The object of this paper is to evaluate convolution sums involving three divisor functions. As mentioned earlier, we call them as triple convolution sums, which are defined by

$$W_{d_1, d_2, d_3}^{k_1-1, k_2-1, k_3-1}(n) = \sum_{\substack{l_1, l_2, l_3 \in \mathbb{N}, \\ d_1l_1 + d_2l_2 + d_3l_3 = n}} \sigma_{k_1-1}(l_1)\sigma_{k_2-1}(l_2)\sigma_{k_3-1}(l_3),$$

where k_i 's are even positive integers and d_1, d_2, d_3 are relatively prime positive integers. When $k_1 = k_2 = k_3 = 2$, we denote the above convolution sum by $W_{d_1, d_2, d_3}(n)$. The study of these convolution sums was initiated in 1946 by D. B. Lahiri [20]. He explicitly

evaluated the sum $W_{1,1,1}(n)$ in terms of the Fourier coefficients of the Eisenstein series $E_2(z), E_4(z)$ and $E_6(z)$. In 2012, using the Fourier coefficients of these Eisenstein series and the theory of quasimodular forms, Williams et. al. [7], evaluated $W_{1,1,2}(n), W_{1,2,2}(n)$ and $W_{1,2,4}(n)$. In 2016, following the method used by Williams et. al., Yoon Kyung Park [30] extended these results to some extent and evaluated the sums $W_{d_1,d_2,d_3}(n)$ for all positive integers d_1, d_2, d_3, n with $\text{lcm}(d_1, d_2, d_3) \leq 6$. In [31], she further considered the case when $\text{lcm}(a, b, c) = 7, 8$ or 9 . In [35], the first two authors evaluated the triple convolution sums $W_{d_1,d_2,d_3}^{1,3,3}(n)$, where $(d_1, d_2, d_3) \in \{(1, 1, 1), (1, 1, 3), (1, 3, 3), (3, 1, 1), (3, 3, 1)\}$ using the theory of quasimodular forms. They also derived the convolution sums $W_{d_1,d_2,d_3}^{3,3,3}(n)$, where $(d_1, d_2, d_3) \in \{(1, 1, 1), (1, 1, 3), (1, 3, 3)\}$ using the theory of modular forms.

In this paper, we circumvent the use of the methods adopted in the previous works [7, 30, 31, 35] and aim to extend our earlier method developed in [38] to derive explicit evaluation of these triple convolutions sums using the theory of modular forms. Our method, in particular, gives all of these previous results mentioned above. We divide the triple convolution sums into the following four cases:

(i) $k_1 = k_2 = k_3 = 2$, (ii) $k_1 = k_2 = 2$ & $k_3 = k \geq 4$, (iii) $k_1 = 2$ & $k_2, k_3 \geq 4$, and (iv) $k_1, k_2, k_3 \geq 4$, and obtain explicit expressions for the convolution sums $W_{d_1,d_2,d_3}^{k_1-1,k_2-1,k_3-1}(n)$ in terms of the Fourier coefficients of modular forms. At the end, we give several examples to demonstrate our method and also present an application of these convolution sums in determining of formulas for the number of representations of a positive integer by certain positive-definite integral quadratic forms.

This paper is organized as follows. In §2, we give some preliminaries and then state our main results. In §3, we give proofs of our main results. We also derive the necessary double convolution sums which appear in the formulas obtained in our main results. In §4, we provide sample formulas for each of our main theorem. Finally in §5, we provide an example to evaluate representation formulas for certain quadratic forms using the convolution sums studied in our work.

2. PRELIMINARIES AND MAIN RESULTS

For $a, b \in \mathbb{N}$, let $\gcd(a, b)$ and $\text{lcm}(a, b)$ denote the greatest common divisor and least common multiple of a, b respectively. For $k, M \in \mathbb{N}$, let $M_k(M, \chi)$ (resp. $S_k(M, \chi)$) denote the space of modular forms (resp. cusp forms) of weight k , for the congruence subgroup $\Gamma_0(M)$, where χ is a Dirichlet character such that $\chi(-1) = (-1)^k$. When $M = 1$, the space is denoted as M_k (resp. S_k) and when χ is the principal character, the spaces are denoted as $M_k(M)$ and $S_k(M)$ respectively. It is well known that all these spaces are finite-dimensional \mathbb{C} -vector spaces. For an even integer $k \geq 4$, let

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \quad (5)$$

be the Eisenstein series in M_k , where $q = e^{2\pi iz}$ and B_k is the k -th Bernoulli number. For $k = 2$, the Eisenstein series $E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma(n)q^n$ is a quasimodular form of weight 2. For positive integers n, d_1, d_2 with $\gcd(d_1, d_2) = 1$ and for even positive integers k_1, k_2 , we define the following convolution sums:

$$W_{d_1, d_2}^{k_1-1, k_2-1}(n) := \sum_{\substack{l, m \in \mathbb{N} \\ d_1 l + d_2 m = n}} \sigma_{k_1-1}(l) \sigma_{k_2-1}(m), \quad (6)$$

$$W_{\widetilde{d}_1, d_2}^{e; k_1-1, k_2-1}(n) := \sum_{\substack{l, m \in \mathbb{N} \\ d_1 l + d_2 m = n}} l^e \sigma_{k_1-1}(l) \sigma_{k_2-1}(m), \quad (7)$$

$$W_{d_1, \widetilde{d}_2}^{e; k_1-1, k_2-1}(n) := \sum_{\substack{l, m \in \mathbb{N} \\ d_1 l + d_2 m = n}} m^e \sigma_{k_1-1}(l) \sigma_{k_2-1}(m). \quad (8)$$

If $k_1 = k_2 = 2$, the sum in (6) is denoted as $W_{d_1, d_2}(n)$. When $e = 1, k_1, k_2 \geq 2$, we remove the superscript e and simply denote the last two sums by $W_{\widetilde{d}_1, d_2}^{k_1-1, k_2-1}(n)$ and $W_{d_1, \widetilde{d}_2}^{k_1-1, k_2-1}(n)$ respectively. When $k_1 = k_2 = 2$ and $e = 1$, we denote the sums defined in (7), (8) by $W_{\widetilde{d}_1, d_2}(n)$ and $W_{d_1, \widetilde{d}_2}(n)$ respectively. Further, when $e = 1$, it is easy to verify the following relation between $W_{\widetilde{d}_1, d_2}^{k_1-1, k_2-1}(n)$ and $W_{d_1, \widetilde{d}_2}^{k_1-1, k_2-1}(n)$:

$$d_1 W_{\widetilde{d}_1, d_2}^{k_1-1, k_2-1}(n) + d_2 W_{d_1, \widetilde{d}_2}^{k_1-1, k_2-1}(n) = n W_{d_1, d_2}^{k_1-1, k_2-1}(n). \quad (9)$$

We now state the main results of this paper. The following notations are used in our theorems. For positive integers d_1, d_2, d_3 with $\gcd(d_1, d_2, d_3) = 1$, let $d = \text{lcm}(d_1, d_2, d_3)$ and further, we denote by $\lambda_k(d)$, the dimension of the finite dimensional vector space $M_k(d)$. The double convolution sums $W_{d_1, d_2}^{k_1-1, k_2-1}(n)$, $W_{\widetilde{d}_1, d_2}^{e; k_1-1, k_2-1}(n)$ and $W_{d_1, \widetilde{d}_2}^{e; k_1-1, k_2-1}(n)$ appearing in the following four theorems are defined respectively by (6), (7) and (8).

Theorem 2.1. *For positive integers n, d_1, d_2, d_3 such that $\gcd(d_1, d_2, d_3) = 1$, we have*

$$\begin{aligned} W_{d_1, d_2, d_3}(n) &= -\frac{1}{24^2} \sigma\left(\frac{n}{d_1}\right) + \frac{1}{24^2} \left(-\frac{18n^2}{d_1 d_3} + \frac{3n}{d_1} + \frac{6n}{d_3} - 1 \right) \sigma\left(\frac{n}{d_2}\right) + \frac{1}{24} \left(1 - \frac{6n}{d_3} \right) W_{d_1, d_2}(n) \\ &\quad + \frac{1}{24^2} \left(-\frac{18n^2}{d_1 d_2} + \frac{3n}{d_1} + \frac{6n}{d_2} - 1 \right) \sigma\left(\frac{n}{d_3}\right) + \frac{1}{24} \left(1 - \frac{3n}{d_1} \right) W_{d_2, d_3}(n) \\ &\quad + \frac{1}{24} \left(1 - \frac{6n}{d_2} \right) W_{d_1, d_3}(n) + \frac{d_1}{4d_3} W_{\widetilde{d}_1, d_2}(n) + \frac{d_1}{4d_2} W_{d_1, \widetilde{d}_3}(n) + \frac{1}{24^3} \sum_{i=1}^{\lambda_6(d)} \alpha_i a_{f_i}(n), \end{aligned}$$

where $a_{f_i}(n)$ denotes the n -th Fourier coefficient of the i -th basis element $f_i(z)$, $1 \leq i \leq \lambda_6(d)$ of a basis for the vector space $M_6(d)$, $d = \text{lcm}(d_1, d_2, d_3)$.

Theorem 2.2. Let n, k, d_1, d_2, d_3 be positive integers with $k (\geq 4)$ even and $\gcd(d_1, d_2, d_3) = 1$. Then, we have

$$\begin{aligned} W_{d_1, d_2, d_3}^{1, 1, k-1}(n) &= \frac{B_k}{48k} \left(\frac{6n}{d_2} - 1 \right) \sigma \left(\frac{n}{d_1} \right) + \frac{B_k}{48k} \left(\frac{6n}{d_1} - 1 \right) \sigma \left(\frac{n}{d_2} \right) - \frac{1}{24^2} \sigma_{k-1} \left(\frac{n}{d_3} \right) + \frac{B_k}{2k} W_{d_1, d_2}(n) \\ &\quad + \frac{1}{24} W_{d_1, d_3}^{1, k-1}(n) + \frac{1}{24} W_{d_2, d_3}^{1, k-1}(n) - \frac{d_1}{4d_2} W_{\widetilde{d}_1, d_3}^{1, k-1}(n) - \frac{d_2}{4d_1} W_{\widetilde{d}_2, d_3}^{1, k-1}(n) \\ &\quad + \frac{B_k}{24 \times 48k} \sum_{i=1}^{\lambda_{k+4}(d)} \beta_i a_{g_i}(n), \end{aligned}$$

where $a_{g_i}(n)$ denotes the n -th Fourier coefficient of the i -th basis element $g_i(z)$, $1 \leq i \leq \lambda_{k+4}(d)$ of a basis for the vector space $M_{k+4}(d)$, $d = \text{lcm}(d_1, d_2, d_3)$.

Theorem 2.3. For positive integers n, d_1, d_2, d_3 with $\gcd(d_1, d_2, d_3) = 1$ and even positive integers $k_2, k_3 \geq 4$, we have

$$\begin{aligned} W_{d_1, d_2, d_3}^{1, k_2-1, k_3-1}(n) &= -\frac{B_{k_2} B_{k_3}}{4k_2 k_3} \sigma \left(\frac{n}{d_1} \right) + \frac{B_{k_3}}{4k_3} \left(\frac{n}{(k_2 + k_3)d_1} - \frac{1}{12} \right) \sigma_{k_2-1} \left(\frac{n}{d_2} \right) - \frac{B_{k_2}}{48k_2} \sigma_{k_3-1} \left(\frac{n}{d_3} \right) \\ &\quad + \frac{B_{k_2} n}{4k_2(k_2 + k_3)d_1} \sigma_{k_3-1} \left(\frac{n}{d_3} \right) + \frac{B_{k_3}}{2k_3} W_{d_1, d_2}^{1, k_2-1}(n) + \frac{B_{k_2}}{2k_2} W_{d_1, d_3}^{1, k_3-1}(n) \\ &\quad + \left(\frac{1}{24} - \frac{n}{2(k_2 + k_3)d_1} \right) W_{d_2, d_3}^{k_2-1, k_3-1}(n) + \frac{B_{k_2} B_{k_3}}{8(k_2 + k_3)k_2 k_3 d_1} \sum_{i=1}^{\lambda_{k_2+k_3+2}(d)} \gamma_i a_{h_i}(n), \end{aligned}$$

where $a_{h_i}(n)$ denotes the n -th Fourier coefficient of the i -th basis element $h_i(z)$, $1 \leq i \leq \lambda_{k_2+k_3+2}(d)$ of the space $M_{k_2+k_3+2}(d)$, $d = \text{lcm}(d_1, d_2, d_3)$.

Theorem 2.4. For positive integers n, d_1, d_2, d_3 with $\gcd(d_1, d_2, d_3) = 1$ and even positive integers $k_1, k_2, k_3 \geq 4$, we have

$$\begin{aligned} W_{d_1, d_2, d_3}^{k_1-1, k_2-1, k_3-1}(n) &= -\frac{B_{k_1} B_{k_2} B_{k_3}}{8k_1 k_2 k_3} \sum_{j=1}^3 \frac{2k_j}{B_{k_j}} \sigma_{k_i-1} \left(\frac{n}{d_j} \right) + \frac{B_{k_3}}{2k_3} W_{d_1, d_2}^{k_1-1, k_2-1}(n) + \frac{B_{k_2}}{2k_2} W_{d_1, d_3}^{k_1-1, k_3-1}(n) \\ &\quad + \frac{B_{k_1}}{2k_1} W_{d_2, d_3}^{k_2-1, k_3-1}(n) - \frac{B_{k_1} B_{k_2} B_{k_3}}{8k_1 k_2 k_3} \sum_{i=1}^{\lambda_{k_1+k_2+k_3}(d)} \delta_i a_{w_i}(n), \end{aligned}$$

where $a_{w_i}(n)$ denotes the n -th Fourier coefficient of the i -th basis element $w_i(z)$, $1 \leq i \leq \lambda_{k_1+k_2+k_3}(d)$ of the space $M_{k_1+k_2+k_3}(d)$, $d = \text{lcm}(d_1, d_2, d_3)$.

Remark 2.5. As mentioned earlier, the (double) convolution sums appearing in all the above four theorems are defined by (6), (7), (8). The sum in (8) follows from the relation (9) (using (6), (7)). Therefore, it is enough to compute the two convolution sums (6) and (7) (with $e = 1$). All these convolution sums (except for the cases $k_1, k_2 \geq 4$ for (7) and $k_1, k_2 \geq 4$ with $e = 1$ for (8)) have been obtained in our earlier work [38, Corollaries 2.2,

2.3, Theorems 3.2, 3.3], which we present here. Let d_1, d_2 be positive integers which are relatively prime and let $k \geq 4$ is an even integer. Then, we have for $n \in \mathbb{N}$,

$$W_{d_1, d_2}^{1, k-1}(n) = \frac{B_k}{2k} \sigma(n/d_1) + \left(\frac{1}{24} - \frac{n}{2kd_1} \right) \sigma_{k-1}(n/d_2) - \frac{B_k}{4d_1 k^2} \sum_{i=1}^{\lambda_{k+2}(d_1 d_2)} \alpha'_i A_i(n), \quad (10)$$

$$W_{d_1, d_2}(n) = \frac{1}{24} \left(1 - \frac{6n}{d_2} \right) \sigma(n/d_1) + \frac{1}{24} \left(1 - \frac{6n}{d_1} \right) \sigma(n/d_2) - \frac{1}{576} \sum_{i=1}^{\lambda_4(d_1 d_2)} \beta'_i B_i(n), \quad (11)$$

$$\begin{aligned} W_{\tilde{d}_1, d_2}^{1, k-1}(n) &= \frac{6n^2 - d_1(k+1)n}{12d_1^2(k+1)(k+2)} \sigma_{k-1}(n/d_2) + \frac{B_k}{2d_1(k+2)} n \sigma(n/d_1) \\ &\quad + \frac{2n}{d_1(k+2)} W_{d_1, d_2}^{1, k-1}(n) + \frac{B_k}{4d_1^2 k(k+1)(k+2)} \sum_{i=1}^{\lambda_{k+4}(d_1 d_2)} \gamma'_i C_i(n), \end{aligned} \quad (12)$$

$$\begin{aligned} W_{\tilde{d}_1, d_2}(n) &= \frac{1}{144d_1} \left(\frac{6}{d_2} n^2 + 3n - \frac{2d_1 n}{d_2} \right) \sigma(n/d_1) + \frac{1}{144d_1} \left(\frac{6}{d_1} n^2 - 3n + \frac{d_1}{3} \right) \sigma(n/d_2) \\ &\quad - \frac{5}{216} \sigma_3(n/d_1) + \frac{n}{2d_1} W_{d_1, d_2}(n) + \frac{5}{9} W_{d_2, d_1}^{1, 3}(n) + \frac{d_1}{3d_2} W_{\tilde{1}, 1}(n/d_1) \\ &\quad + \frac{1}{6 \times 576d_1} \sum_{i=1}^{\lambda_6(d_1 d_2)} \delta'_i F_i(n). \end{aligned} \quad (13)$$

In the above, $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i$ are constants and the sum $W_{\tilde{1}, 1}(n)$ is given by

$$W_{\tilde{1}, 1}(n) = \sum_{l+m=n} l \sigma(l) \sigma(m) = \frac{5}{24} n \sigma_3(n) - \frac{1}{24} (6n^2 - n) \sigma(n). \quad (14)$$

Further, $A_i(n), B_i(n), C_i(n)$ and $F_i(n)$ are the n -th Fourier coefficients of basis elements of the spaces $M_{k+2}(d_1 d_2)$, $M_4(d_1 d_2)$, $M_{k+4}(d_1 d_2)$ and $M_6(d_1 d_2)$ respectively. The remaining two convolution sums are derived in the next section (Lemma 3.1).

3. PROOFS

We first prove the remaining two convolution sums (mentioned in Remark 2.5) needed for the determination of the required formulas in our theorems.

Lemma 3.1. *Let n, d_1, d_2 be positive integers with $\gcd(d_1, d_2) = 1$ and $k_1, k_2 \geq 4$ be even integers. Then there exist constants ν_i and ϵ_i such that*

$$W_{d_1, d_2}^{k_1-1, k_2-1}(n) = \frac{B_{k_2}}{2k_2} \sigma_{k_1-1}(n/d_1) + \frac{B_{k_1}}{2k_1} \sigma_{k_2-1}(n/d_2) + \frac{B_{k_1} B_{k_2}}{4k_1 k_2} \sum_{i=1}^{\lambda_{k_1+k_2}(d_1 d_2)} \nu_i a_{\psi_i}(n), \quad (15)$$

where $a_{\psi_i}(n)$ denotes the n -th Fourier coefficient of the i -th basis element $\psi_i(z)$, $1 \leq i \leq \lambda_{k_1+k_2}(d_1 d_2)$ of the space $M_{k_1+k_2}(d_1 d_2)$, and

$$\begin{aligned} W_{d_1, d_2}^{k_1-1, k_2-1}(n) &= \frac{B_{k_2} n}{2(k_1 + k_2)d_1} \sigma_{k_1-1}(n/d_1) - \frac{B_{k_1} n}{2(k_1 + k_2)d_1} \sigma_{k_2-1}(n/d_2) \\ &\quad + \frac{k_1 n}{(k_1 + k_2)d_1} W_{d_1, d_2}^{k_1-1, k_2-1}(n) + \frac{B_{k_1} B_{k_2}}{4k_1 k_2 (k_1 + k_2)d_1} \sum_{i=1}^{\lambda_{k_1+k_2+2}(d_1 d_2)} \epsilon_i a_{\Psi_i}(n). \end{aligned} \quad (16)$$

where $a_{\Psi_i}(n)$ denotes the n -th Fourier coefficient of the i -th basis element $\Psi_i(z)$, $1 \leq i \leq \lambda_{k_1+k_2+2}(d_1 d_2)$ of the space $M_{k_1+k_2+2}(d_1 d_2)$.

Proof. The first formula follows by expressing the modular forms $E_{k_1}(d_1 z)E_{k_2}(d_2 z)$ in terms of a basis (say $\{\psi_i(z) : 1 \leq i \leq \lambda_{k_1+k_2}(d_1 d_2)\}$) of the space $M_{k_1+k_2}(d_1 d_2)$ and then comparing the n -th Fourier coefficient on both the sides.

We now give a proof for the second sum. Using the extended Ramanujan-Serre derivative map (given by (4)) (whose mapping property was obtained in our earlier work [38]), we see that

$$\begin{aligned} \vartheta_{k_1; (d_2, d_1)}(E_{k_1}(z)) &= d_1 D E_{k_1}(d_1 z) - \frac{k_1}{12} d_2 E_2(d_2 z) E_{k_1}(d_1 z), \\ \vartheta_{k_2; (d_2, d_1)}(E_{k_2}(z)) &= d_2 D E_{k_2}(d_2 z) - \frac{k_2}{12} d_1 E_2(d_1 z) E_{k_2}(d_2 z) \end{aligned}$$

are modular forms of weight $k_1 + 2$, level $d_1 d_2$ and $k_2 + 2$, level d_2 respectively. Therefore, multiplying the first one by $k_2 E_{k_2}(d_2 z)$ and the second one by $k_1 E_{k_1}(d_1 z)$ (note that after this multiplication, both are modular forms of weight $k_1 + k_2 + 2$ with level $d_1 d_2$) and subtracting them, it follows that the function

$$k_2 d_1 D E_{k_1}(d_1 z) E_{k_2}(d_2 z) - k_1 d_2 D E_{k_2}(d_2 z) E_{k_1}(d_1 z)$$

is a modular form of weight $k_1 + k_2 + 2$ and level $d_1 d_2$. Now, by choosing a suitable basis for the space $M_{k_1+k_2+2}(d_1 d_2)$, say $\{\Psi_i(z) : 1 \leq i \leq \lambda_{k_1+k_2+2}(d_1 d_2)\}$, we express the above modular form as a linear combination of these basis elements. Therefore, denoting the n -th Fourier coefficient of $\Psi_i(z)$ by $a_{\Psi_i}(n)$, there exist constants ϵ_i such that

$$k_2 d_1 D E_{k_1}(d_1 z) E_{k_2}(d_2 z) - k_1 d_2 D E_{k_2}(d_2 z) E_{k_1}(d_1 z) = \sum_{i=1}^{\lambda_{k_1+k_2+2}(d_1 d_2)} \epsilon_i \Psi_i(z).$$

Comparing the n -th Fourier coefficient on both sides and using the relation (9) between the convolution sums we get, after simplifications, the following expression.

$$\begin{aligned} -\frac{2k_1 k_2 n}{B_{k_1}} \sigma_{k_1-1}(n/d_1) + \frac{2k_1 k_2 n}{B_{k_2}} \sigma_{k_2-1}(n/d_2) - \frac{4k_1^2 k_2 n}{B_{k_1} B_{k_2} d_2} W_{d_1, d_2}^{k_1-1, k_2-1}(n) \\ + \frac{4k_1 k_2 (k_1 + k_2) d_1}{B_{k_1} B_{k_2}} W_{d_1, d_2}^{k_1-1, k_2-1}(n) = \sum_{i=1}^{\lambda_{k_1+k_2+2}(d_1 d_2)} \epsilon_i a_{\Psi_i}(n). \end{aligned}$$

This completes the proof. \square

3.1. Proof of Theorem 2.1. Let d_1, d_2, d_3 be positive integers such that $\gcd(d_1, d_2, d_3) = 1$ and let $N_1 = \text{lcm}(d_2, d_3)$. Denote the function given by (3) by $F_{d_1, d_2}(z)$. Then we see that

$$F_{d_2, d_3}(z) = \frac{6d_2}{d_3} DE_2(d_2 z) + \frac{6d_3}{d_2} DE_2(d_3 z) - E_2(d_2 z) E_2(d_3 z)$$

is a modular form belonging to $M_4(N_1)$. Let $N = \text{lcm}(d_1, d_2, d_3)$. Then applying the theta operator $\vartheta_{4;(d_1, 1)}$ given by (4) on $F_{d_2, d_3}(z)$, we get the following modular form in $M_6(N)$:

$$\begin{aligned} \vartheta_{4;(d_1, 1)}(F_{d_2, d_3}(z)) &= D(F_{d_2, d_3}(z)) - \frac{4}{12} d_1 E_2(d_1 z)(F_{d_2, d_3}(z)) \\ &= D\left(\frac{6d_2}{d_3} DE_2(d_2 z) + \frac{6d_3}{d_2} DE_2(d_3 z) - E_2(d_2 z) E_2(d_3 z)\right) \\ &\quad - \frac{d_1}{3} E_2(d_1 z) \left(\frac{6d_2}{d_3} DE_2(d_2 z) + \frac{6d_3}{d_2} DE_2(d_3 z) - E_2(d_2 z) E_2(d_3 z)\right) \\ &= \frac{6d_2^2}{d_3} D^2 E_2(d_2 z) + \frac{6d_3^2}{d_2} D^2 E_2(d_3 z) - d_2 D E_2(d_2 z) E_2(d_3 z) - d_3 E_2(d_2 z) D E_2(d_3 z) \\ &\quad - \frac{2d_1 d_2}{d_3} E_2(d_1 z) D E_2(d_2 z) - \frac{2d_1 d_3}{d_2} E_2(d_1 z) D E_2(d_3 z) + \frac{d_1}{3} E_2(d_1 z) E_2(d_2 z) E_2(d_3 z). \end{aligned}$$

We first multiply this modular form by $\frac{3}{d_1}$ and then express it as a linear combination of a basis for the space $M_6(N)$. Now comparing the n -th Fourier coefficients of both the sides of this expression, we obtain the required convolution sum given in the theorem.

Remark 3.2. Replacing (d_1, d_2, d_3) by (d_2, d_3, d_1) and (d_3, d_1, d_2) in the above proof, we get the following two modular forms, which also belong to $M_6(N)$.

- $E_2(d_2 z) E_2(d_3 z) E_2(d_1 z) + \frac{18d_3^2}{d_1 d_2} D^2 E_2(d_3 z) + \frac{18d_1^2}{d_2 d_3} D^2 E_2(d_1 z) - \frac{3d_3}{d_2} D E_2(d_3 z) E_2(d_1 z) - \frac{3d_1}{d_2} E_2(d_3 z) D E_2(d_1 z) - \frac{6d_3}{d_1} E_2(d_2 z) D E_2(d_3 z) - \frac{6d_1}{d_3} E_2(d_2 z) D E_2(d_1 z).$
- $E_2(d_3 z) E_2(d_1 z) E_2(d_2 z) + \frac{18d_1^2}{d_2 d_3} D^2 E_2(d_1 z) + \frac{18d_2^2}{d_1 d_3} D^2 E_2(d_2 z) - \frac{3d_1}{d_3} D E_2(d_1 z) E_2(d_2 z) - \frac{3d_2}{d_3} E_2(d_1 z) D E_2(d_2 z) - \frac{6d_1}{d_2} E_2(d_3 z) D E_2(d_1 z) - \frac{6d_2}{d_1} E_2(d_3 z) D E_2(d_2 z).$

These two modular forms give rise to two different expressions for the convolution sum $W_{d_1, d_2, d_3}(n)$. Now using these three modular forms (the above two along with the modular form in the proof of the theorem), we obtain the following unique modular form (symmetric w.r.t. d_1, d_2 and d_3) in the space $M_6(N)$:

$$\begin{aligned} E_2(d_1z)E_2(d_2z)E_2(d_3z) + \frac{12d_1^2}{d_2d_3}D^2E_2(d_1z) + \frac{12d_2^2}{d_3d_1}D^2E_2(d_2z) + \frac{12d_3^2}{d_1d_2}D^2E_2(d_3z) \\ - \frac{9d_1}{2d_3}DE_2(d_1z)E_2(d_2z) - \frac{9d_2}{2d_1}DE_2(d_2z)E_2(d_3z) - \frac{9d_3}{2d_2}DE_2(d_3z)E_2(d_1z) \\ - \frac{9d_1}{2d_2}DE_2(d_1z)E_2(d_3z) - \frac{9d_2}{2d_3}DE_2(d_2z)E_2(d_1z) - \frac{9d_3}{2d_1}DE_2(d_3z)E_2(d_2z). \end{aligned}$$

One can also use this modular form to get the convolution sum $W_{d_1, d_2, d_3}(n)$.

3.2. Proof of Theorem 2.2. Let the notations be as in the proof of Theorem 2.1. Multiplying the modular form $F_{d_1, d_2}(z) \in M_4(N_1)$ by $E_k(d_3z)$, we see that the function

$$\frac{6d_1}{d_2}DE_2(d_1z)E_k(d_3z) + \frac{6d_2}{d_1}DE_2(d_2z)E_k(d_3z) - E_2(d_1z)E_2(d_2z)E_k(d_3z)$$

is a modular form in $M_6(N)$. Therefore, expressing it in terms of a basis of $M_6(N)$ and comparing the n -th Fourier coefficients, the required convolution sum follows.

3.3. Proof of Theorem 2.3. In this case, we consider the modular form $E_{k_2}(d_2z)E_{k_3}(d_3z)$ of weight $k_2 + k_3$ and level $\text{lcm}(d_2, d_3)$. Then we apply the theta operator (refer Eq. (4)) $\vartheta_{k_2+k_3; (d_1, 1)}$ on the above modular form of weight $k_2 + k_3$ to obtain the following modular form:

$$d_2DE_{k_2}(d_2z)E_{k_3}(d_3z) + d_3E_{k_2}(d_2z)DE_{k_3}(d_3z) - \frac{k_2 + k_3}{12}d_1E_2(d_1z)E_{k_2}(d_2z)E_{k_3}(d_3z),$$

which belongs to the space $M_{k_2+k_3+2}(N)$, where $N = \text{lcm}(d_1, d_2, d_3)$. As before, we express this modular form as a linear combination of a basis for $M_{k_2+k_3+2}(N)$, and comparing the n -th Fourier coefficients gives the convolution sum $W_{d_1, d_2, d_3}^{1, k_2-1, k_3-1}(n)$.

3.4. Proof of Theorem 2.4. Proof of this theorem is straightforward. It is clear that the product of Eisenstein series $E_{k_1}(d_1z)E_{k_2}(d_2z)E_{k_3}(d_3z)$ is a modular form of weight $k_1 + k_2 + k_3$, for the congruence subgroup $\Gamma_0(N)$, $N = \text{lcm}(d_1, d_2, d_3)$. Writing this modular form in terms of a basis of $M_{k_1+k_2+k_3}(N)$ and comparing the n -th Fourier coefficients, we obtain the convolution sum $W_{d_1, d_2, d_3}^{k_1-1, k_2-1, k_3-1}(n)$, as stated in the theorem.

4. SAMPLE FORMULAS

In this section, we shall give some examples and provide formulas for certain cases. Some formulas for (15) (obtained in Lemma 3.1) are provided in the next section. As the purpose of this article is to show that explicit basis for the space of modular forms of weight k and level N would give all the convolution sums studied in this work, we shall be providing explicit basis for the space of modular forms of some specific weight and level and use them to get our sample formulas. We shall be presenting these bases in the Appendix section. A typical basis in our construction consists of Eisenstein series and their duplications and newforms and their duplications. A newform of weight k , level N is denoted as $\Delta_{k,N}(z)$, whose Fourier coefficients are denoted as $\tau_{k,N}(n)$, if the newform is unique. If there are more than one newform of particular weight k and level N , then we denote them by $\Delta_{k,N;j}(z)$ and the respective Fourier coefficient by $\tau_{k,N;j}(n)$, where $j \geq 1$.

We list below two tables which give explicit bases for modular forms of weights $k = 6, 8, 10, 12$ and levels $2, 3, 6$. Explicit expressions of newforms used in these bases are given in the Appendix section.

Table 1.

Space	Basis	Space	Basis
$M_6(2)$	$\{E_6(z), E_6(2z)\}$	$M_6(3)$	$\{E_6(z), E_6(2z), \Delta_{6,3}(z)\}$
$M_8(2)$	$\{E_8(z), E_8(2z), \Delta_{8,2}(z)\}$	$M_8(3)$	$\{E_8(z), E_8(3z), \Delta_{8,3}(z)\}$
$M_{10}(2)$	$\{E_{10}(z), E_{10}(2z), \Delta_{10,2}(z)\}$	$M_{10}(3)$	$\{E_{10}(z), E_{10}(3z), \Delta_{10,3;1}(z), \Delta_{10,3;2}(z)\}$

Table 2.

Space	Basis
$M_{10}(6)$	$M_{10}(2) \cup M_{10}(3) \cup \{E_{10}(6z), \Delta_{10,2}(3z), \Delta_{10,3;1}(2z), \Delta_{10,3;2}(2z), \Delta_{10,6}(z)\}$
$M_{12}(6)$	$\{E_{12}(dz), \Delta(dz); d 6, \Delta_{12,3}(z), \Delta_{12,3}(2z), \Delta_{12,6;1}(z), \Delta_{12,6;2}(z), \Delta_{12,6;3}(z)\}$

4.1. Sample formulas for the convolutions sums $W_{d_1,d_2}^{3,3}(n)$. Here we give examples for the cases $(d_1, d_2) = (1, 2), (1, 3), (1, 6), (2, 3)$, for which we use a basis for the space $M_{10}(2)$ (for the first case), $M_{10}(3)$ (for the second case) and $M_{10}(6)$ (for the last two cases), which are given in Tables 1 and 2.

$$\begin{aligned} W_{1,2}^{3,3}(n) &= \frac{n}{4080}\sigma_7(n) + \frac{n}{255}\sigma_7\left(\frac{n}{2}\right) - \frac{n}{240}\sigma_3(n) + \frac{n}{544}\tau_{8,2}(n) + \frac{1}{480}\tau_{10,2}(n), \\ W_{1,3}^{3,3}(n) &= \frac{n}{19680}\sigma_7(n) + \frac{81n}{19680}\sigma_7\left(\frac{n}{3}\right) - \frac{n}{240}\sigma_3(n) + \frac{n}{492}\tau_{8,3}(n) + \frac{1}{480}\tau_{10,3;2}(n), \end{aligned}$$

$$\begin{aligned}
W_{\tilde{1},6}^{3,3}(n) &= \frac{n}{334560}\sigma_7(n) + \frac{n}{20910}\sigma_7\left(\frac{n}{2}\right) + \frac{27n}{111520}\sigma_7\left(\frac{n}{3}\right) + \frac{27n}{6970}\sigma_7\left(\frac{n}{6}\right) - \frac{n}{240}\sigma_3(n) \\
&\quad + \frac{7n}{16320}\tau_{8,2}(n) + \frac{189n}{5440}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{n}{1640}\tau_{8,3}(n) + \frac{2n}{205}\tau_{8,3}\left(\frac{n}{2}\right) + \frac{n}{960}\tau_{8,6}(n) \\
&\quad + \frac{1}{960}\tau_{10,2}(n) + \frac{81}{320}\tau_{10,2}\left(\frac{n}{3}\right) + \frac{1}{1760}\tau_{10,3;2}(n) + \frac{1}{55}\tau_{10,3;2}\left(\frac{n}{2}\right) + \frac{1}{2112}\tau_{10,6}(n), \\
W_{\tilde{2},3}^{3,3}(n) &= \frac{n}{669120}\sigma_7(n) + \frac{n}{41820}\sigma_7\left(\frac{n}{2}\right) + \frac{27n}{223040}\sigma_7\left(\frac{n}{3}\right) + \frac{27n}{13940}\sigma_7\left(\frac{n}{6}\right) - \frac{n}{480}\sigma_3\left(\frac{n}{2}\right) \\
&\quad + \frac{7n}{32640}\tau_{8,2}(n) + \frac{189n}{10880}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{n}{3280}\tau_{8,3}(n) + \frac{n}{205}\tau_{8,3}\left(\frac{n}{2}\right) - \frac{n}{1920}\tau_{8,6}(n) \\
&\quad - \frac{1}{1920}\tau_{10,2}(n) - \frac{81}{640}\tau_{10,2}\left(\frac{n}{3}\right) + \frac{1}{3520}\tau_{10,3;2}(n) + \frac{1}{110}\tau_{10,3;2}\left(\frac{n}{2}\right) + \frac{1}{4224}\tau_{10,6}(n).
\end{aligned}$$

In the next section, we will be providing examples for our four theorems for the cases $(d_1, d_2, d_3) \in \{(1, 1, 1), (1, 1, 2), (1, 2, 2), (1, 1, 3), (1, 3, 3)\}$. The corresponding levels for these cases are: 1, 2, 2, 3, 3, respectively. Theorem 2.1 corresponds to the case $k_i = 2$, $i = 1, 2, 3$ and so the space of modular forms has weight 6. For Theorem 2.2, we give examples when $k = 4$, for Theorem 2.3 we take $k_2 = k_3 = 4$ and for Theorem 2.4, we take $k_1 = k_2 = k_3 = 4$. The respective bases for the spaces of modular forms (of weights 6, 8, 10 and 12) are presented in Tables 1 and 2. For level 1, the space M_k is generated by $E_k(z)$ when $k = 6, 8, 10$ and the space M_{12} is spanned by the forms $E_{12}(z)$ and the Ramanujan delta function $\Delta(z) = \sum_{n \geq 1} \tau(n)e^{2\pi i n z}$.

4.2. Sample formulas for Theorems 2.1, 2.2, 2.3, 2.4.

Examples for Theorem 2.1.

Using the basis for the spaces $M_6(N)$, $N = 2, 3$ as given in Table 1, we obtain the following formulas for $n \in \mathbb{N}$.

$$\begin{aligned}
W_{1,1,1}(n) &= \frac{7}{192}\sigma_5(n) + \frac{5 - 15n}{96}\sigma_3(n) + \frac{24n^2 - 12n + 1}{192}\sigma(n), \\
W_{1,1,2}(n) &= \frac{5}{576}\sigma_5(n) + \frac{1}{36}\sigma_5\left(\frac{n}{2}\right) + \frac{14 - 27n}{576}\sigma_3(n) + \frac{1 - 3n}{36}\sigma_3\left(\frac{n}{2}\right) + \frac{12n^2 - 9n + 1}{288}\sigma(n) \\
&\quad + \frac{24n^2 - 12n + 1}{576}\sigma\left(\frac{n}{2}\right), \\
W_{1,2,2}(n) &= \frac{5}{576}\sigma_5(n) + \frac{5}{144}\sigma_5\left(\frac{n}{2}\right) + \frac{2 - 3n}{288}\sigma_3(n) + \frac{13 - 27n}{288}\sigma_3\left(\frac{n}{2}\right) + \frac{6n^2 - 6n + 1}{576}\sigma(n) \\
&\quad + \frac{12n^2 - 9n + 1}{288}\sigma\left(\frac{n}{2}\right),
\end{aligned}$$

$$\begin{aligned}
W_{1,1,3}(n) &= \frac{5}{1248}\sigma_5(n) + \frac{27}{832}\sigma_5\left(\frac{n}{3}\right) + \frac{3-4n}{144}\sigma_3(n) + \frac{1-3n}{32}\sigma_3\left(\frac{n}{3}\right) + \frac{8n^2-8n+1}{288}\sigma(n) \\
&\quad + \frac{24n^2-12n+1}{576}\sigma\left(\frac{n}{3}\right) - \frac{1}{1872}\tau_{6,3}(n), \\
W_{1,3,3}(n) &= \frac{1}{2496}\sigma_5(n) + \frac{15}{416}\sigma_5\left(\frac{n}{3}\right) + \frac{1-n}{288}\sigma_3(n) + \frac{7-12n}{144}\sigma_3\left(\frac{n}{3}\right) + \frac{8n^2-12n+3}{1728}\sigma(n) \\
&\quad + \frac{8n^2-8n+1}{288}\sigma\left(\frac{n}{3}\right) + \frac{1}{5616}\tau_{6,3}(n).
\end{aligned}$$

Examples for Theorem 2.2.

We use the basis for the space $M_8(N)$, $N = 2, 3$ as given in Table 1 to obtain the following formulas for $n \in \mathbb{N}$.

$$\begin{aligned}
W_{1,1,1}^{1,1,3}(n) &= \frac{1}{288}\sigma_7(n) + \frac{7(1-2n)}{960}\sigma_5(n) + \frac{6n^2-5n}{480}\sigma_3(n) + \frac{6n-1}{480}\sigma(n), \\
W_{1,1,2}^{1,1,3}(n) &= \frac{1}{4896}\sigma_7(n) + \frac{1}{306}\sigma_7\left(\frac{n}{2}\right) + \frac{1-2n}{2880}\sigma_5(n) + \frac{1-2n}{144}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{576}\sigma_3(n) \\
&\quad + \frac{360n^2-30n+5}{2880}\sigma_3\left(\frac{n}{2}\right) + \frac{6n-1}{2880}\sigma(n) + \frac{7}{48960}\tau_{8,2}(n), \\
W_{1,2,2}^{1,1,3}(n) &= \frac{1}{24480}\sigma_7(n) + \frac{7}{2040}\sigma_7\left(\frac{n}{2}\right) + \frac{1-n}{5760}\sigma_5(n) + \frac{41-62n}{5760}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{2880}\sigma_3(n) \\
&\quad + \frac{36n^2-45n+2}{5760}\sigma_3\left(\frac{n}{2}\right) + \frac{3n-1}{5760}\sigma(n) + \frac{6n-1}{5760}\sigma\left(\frac{n}{2}\right) - \frac{1}{24480}\tau_{8,2}(n), \\
W_{1,1,3}^{1,1,3}(n) &= \frac{1}{23616}\sigma_7(n) + \frac{9}{2624}\sigma_7\left(\frac{n}{3}\right) + \frac{(1-2n)}{12480}\sigma_5(n) + \frac{3(1-2n)}{416}\sigma_5\left(\frac{n}{3}\right) - \frac{1}{576}\sigma_3(n) \\
&\quad + \frac{360n^2-30n+5}{2880}\sigma_3\left(\frac{n}{3}\right) + \frac{6n-1}{2880}\sigma(n) + \frac{(1-2n)}{3744}\tau_{6,3}(n) + \frac{1}{3280}\tau_{8,3}(n), \\
W_{1,3,3}^{1,1,3}(n) &= \frac{1}{236160}\sigma_7(n) + \frac{91}{26240}\sigma_7\left(\frac{n}{3}\right) + \frac{3-2n}{74880}\sigma_5(n) + \frac{181-242n}{24960}\sigma_5\left(\frac{n}{3}\right) - \frac{1}{5760}\sigma_3(n) \\
&\quad + \frac{24n^2-40n+1}{5760}\sigma_3\left(\frac{n}{3}\right) + \frac{2n-1}{5760}\sigma(n) + \frac{6n-1}{5760}\sigma\left(\frac{n}{3}\right) + \frac{3-2n}{22464}\tau_{6,3}(n) - \frac{11}{177120}\tau_{8,3}(n).
\end{aligned}$$

Examples for Theorem 2.3.

In this case, we use the basis for the space $M_{10}(N)$, $N = 2, 3$ as given in Table 1 to obtain the following formulas for $n \in \mathbb{N}$.

$$W_{1,1,1}^{1,3,3}(n) = \frac{11}{57600}\sigma_9(n) + \frac{2-3n}{5760}\sigma_7(n) - \frac{7}{9600}\sigma_5(n) + \frac{(3n-1)}{2880}\sigma_3(n) + \frac{1}{57600}\sigma(n),$$

$$\begin{aligned}
W_{1,1,2}^{1,3,3}(n) &= \frac{7}{595200}\sigma_9(n) + \frac{1}{5580}\sigma_9\left(\frac{n}{2}\right) + \frac{2-3n}{97920}\sigma_7(n) + \frac{2-3n}{6120}\sigma_7\left(\frac{n}{2}\right) - \frac{11}{28800}\sigma_5(n) \\
&\quad - \frac{1}{2880}\sigma_5\left(\frac{n}{2}\right) + \frac{3n-1}{5760}\sigma_3(n) + \frac{(3n-1)}{5760}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{57600}\sigma(n) \\
&\quad + \frac{2-3n}{13056}\tau_{8,2}(n) + \frac{11}{119040}\tau_{10,2}(n), \\
W_{1,2,2}^{1,3,3}(n) &= \frac{1}{1785600}\sigma_9(n) + \frac{17}{89280}\sigma_9\left(\frac{n}{2}\right) + \frac{2-3n}{5760}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{28800}\sigma_5(n) - \frac{1}{1440}\sigma_5\left(\frac{n}{2}\right) \\
&\quad + \frac{3n-1}{2880}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{57600}\sigma(n) + \frac{1}{59520}\tau_{10,2}(n), \\
W_{1,1,3}^{1,3,3}(n) &= \frac{91}{38649600}\sigma_9(n) + \frac{81}{429440}\sigma_9\left(\frac{n}{3}\right) + \frac{2-3n}{472320}\sigma_7(n) + \frac{9(2-3n)}{52480}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{23}{62400}\sigma_5(n) - \frac{3}{8320}\sigma_5\left(\frac{n}{3}\right) + \frac{3n-1}{5760}\sigma_3(n) + \frac{3n-1}{5760}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{57600}\sigma(n) \\
&\quad - \frac{1}{74880}\tau_{6,3}(n) + \frac{2-3n}{11808}\tau_{8,3}(n) - \frac{7}{87840}\tau_{10,3;1}(n) + \frac{23}{126720}\tau_{10,3;2}(n), \\
W_{1,3,3}^{1,3,3}(n) &= \frac{1}{38649600}\sigma_9(n) + \frac{41}{214720}\sigma_9\left(\frac{n}{3}\right) + \frac{2-3n}{5760}\sigma_7\left(\frac{n}{3}\right) - \frac{1}{124800}\sigma_5(n) - \frac{3}{4160}\sigma_5\left(\frac{n}{3}\right) \\
&\quad + \frac{(3n-1)}{2880}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{57600}\sigma(n) - \frac{1}{37440}\tau_{6,3}(n) + \frac{1}{197640}\tau_{10,3;1}(n) \\
&\quad + \frac{7}{570240}\tau_{10,3;2}(n).
\end{aligned}$$

Examples for Theorem 2.4.

Finally, by using the basis for the space $M_{12}(N)$, $N = 2, 3$ (which is a subset of the basis of $M_{12}(6)$ given in Table 2), we obtain the following formulas for $n \in \mathbb{N}$.

$$\begin{aligned}
W_{1,1,1}^{3,3,3}(n) &= \frac{1}{19200}\sigma_3(n) - \frac{1}{9600}\sigma_7(n) + \frac{91}{13267200}\sigma_{11}(n) + \frac{1}{22112}\tau(n), \\
W_{1,1,2}^{3,3,3}(n) &= \frac{1}{28800}\sigma_3(n) + \frac{1}{57600}\sigma_3\left(\frac{n}{2}\right) - \frac{19}{489600}\sigma_7(n) - \frac{1}{15300}\sigma_7\left(\frac{n}{2}\right) + \frac{17}{39801600}\sigma_{11}(n) \\
&\quad + \frac{1}{155475}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{32640}\tau_{8,2}(n) + \frac{91}{2653440}\tau(n) + \frac{46}{10365}\tau\left(\frac{n}{2}\right), \\
W_{1,2,2}^{3,3,3}(n) &= \frac{1}{57600}\sigma_3(n) + \frac{1}{28800}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{244800}\sigma_7(n) - \frac{49}{489600}\sigma_7\left(\frac{n}{2}\right) + \frac{1}{39801600}\sigma_{11}(n) \\
&\quad + \frac{17}{2487600}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{32640}\tau_{8,2}(n) + \frac{23}{1326720}\tau(n) + \frac{91}{165840}\tau\left(\frac{n}{2}\right),
\end{aligned}$$

$$\begin{aligned}
W_{1,1,3}^{3,3,3}(n) &= \frac{41}{484252800}\sigma_{11}(n) + \frac{6561}{968505600}\sigma_{11}\left(\frac{n}{3}\right) - \frac{7}{196800}\sigma_7(n) - \frac{9}{131200}\sigma_7\left(\frac{n}{3}\right) \\
&\quad + \frac{1}{28800}\sigma_3(n) + \frac{1}{57600}\sigma_3\left(\frac{n}{3}\right) - \frac{133}{8457840}\tau(n) + \frac{145071}{1879520}\tau\left(\frac{n}{3}\right) \\
&\quad - \frac{1}{29520}\tau_{8,3}(n) + \frac{1}{19856}\tau_{12,3}(n), \\
W_{1,3,3}^{3,3,3}(n) &= \frac{1}{968505600}\sigma_{11}(n) + \frac{1107}{161417600}\sigma_{11}\left(\frac{n}{3}\right) - \frac{1}{1180800}\sigma_7(n) - \frac{61}{590400}\sigma_7\left(\frac{n}{3}\right) \\
&\quad + \frac{1}{57600}\sigma_3(n) + \frac{1}{28800}\sigma_3\left(\frac{n}{3}\right) + \frac{199}{16915680}\tau(n) - \frac{1197}{939760}\tau\left(\frac{n}{3}\right) - \frac{1}{29520}\tau_{8,3}(n) \\
&\quad + \frac{1}{178704}\tau_{12,3}(n).
\end{aligned}$$

5. APPLICATIONS

In this section we demonstrate an application which uses the double and triple convolution sums studied in our work to obtain explicit formulas for the number of representations of a natural number by two positive definite integral quadratic forms of 24 variables. Let \mathbb{Q}_{2k} denote the following quadratic form in $2k$ variables:

$$\mathbb{Q}_{2k} : x_1^2 + x_1x_2 + x_2^2 + \dots + x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2, \quad (17)$$

where $k \geq 1$ is an integer and x_i 's are integers. For $n \in \mathbb{N}$, let $s_{2k}(n)$ denote the number of representation of n by the quadratic form \mathbb{Q}_{2k} . Then it is known that $s_8(n) = 24\sigma_3(n) + 216\sigma_3(n/3)$ (see [23]). Note that $\mathbb{Q}_{24} = \mathbb{Q}_8 \oplus \mathbb{Q}_8 \oplus \mathbb{Q}_8$. Therefore, using the above formula for $s_8(n)$ and the convolution sums $W_{1,1}^{3,3}(n)$, $W_{1,3}^{3,3}(n)$, $W_{1,1,1}^{3,3,3}(n)$, $W_{1,1,3}^{3,3,3}(n)$ and $W_{1,3,3}^{3,3,3}(n)$, in [35], the first two authors obtained the following formula for $s_{24}(n)$:

$$s_{24}(n) = \frac{6552}{50443}\sigma_{11}^*(n) + \frac{402624}{11747}\tau(n) + \frac{293512896}{11747}\tau\left(\frac{n}{3}\right) + \frac{46656}{1241}\tau_{12,3}(n), \quad (18)$$

where $\sigma_{11}^*(n) = \sigma_{11}(n) + 3^6\sigma_{11}\left(\frac{n}{3}\right)$, $\tau(n)$ is the Ramanujan tau function, which is the n -th Fourier coefficient of the unique cusp form of weight 12 for $SL_2(\mathbb{Z})$ and $\tau_{k,N}(n)$ denotes the n -th Fourier coefficient of the normalised newform of weight k and level N .

We adopt similar techniques and use the formula $s_8(n) = 24\sigma_3(n) + 216\sigma_3(n/3)$ to obtain explicit formulas for the following two quadratic forms in 24 variables:

$$\begin{aligned}
\mathcal{Q}_1 &: \mathbb{Q}_8 \oplus \mathbb{Q}_8 \oplus 2\mathbb{Q}_8, \\
\mathcal{Q}_2 &: \mathbb{Q}_8 \oplus 2\mathbb{Q}_8 \oplus 2\mathbb{Q}_8.
\end{aligned}$$

In the following, we give the double and triple convolution sums required to get the formulas for the representation numbers corresponding to the above two quadratic forms.

Using the same approach as given in the proof of Lemma 3.1, in [34], the first two authors obtained explicit formulas for the convolutions sums $W_{1,2}^{3,3}(n)$, $W_{1,3}^{3,3}(n)$, $W_{1,6}^{3,3}(n)$

and $W_{2,3}^{3,3}(n)$, which we list below along with $W_{1,1}^{3,3}(n)$ (which follows from the identity $E_4^2(z) = E_8(z)$):

$$\begin{aligned}
W_{1,1}^{3,3}(n) &= \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n), \\
W_{1,2}^{3,3}(n) &= -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7\left(\frac{n}{2}\right) + \frac{1}{272}\tau_{8,2}(n), \\
W_{1,3}^{3,3}(n) &= -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{9840}\sigma_7(n) + \frac{81}{9840}\sigma_7\left(\frac{n}{3}\right) + \frac{1}{246}\tau_{8,3}(n), \\
W_{2,3}^{3,3}(n) &= -\frac{1}{240}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{167280}\sigma_7(n) + \frac{1}{10455}\sigma_7\left(\frac{n}{2}\right) + \frac{27}{55760}\sigma_7\left(\frac{n}{3}\right) \\
&\quad + \frac{27}{3485}\sigma_7\left(\frac{n}{6}\right) + \frac{7}{8160}\tau_{8,2}(n) + \frac{189}{2720}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{1}{820}\tau_{8,3}(n) + \frac{4}{205}\tau_{8,3}\left(\frac{n}{2}\right) - \frac{1}{480}\tau_{8,6}(n), \\
W_{1,6}^{3,3}(n) &= -\frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{167280}\sigma_7(n) + \frac{1}{10455}\sigma_7\left(\frac{n}{2}\right) + \frac{27}{55760}\sigma_7\left(\frac{n}{3}\right) \\
&\quad + \frac{27}{3485}\sigma_7\left(\frac{n}{6}\right) + \frac{7}{8160}\tau_{8,2}(n) + \frac{189}{2720}\tau_{8,2}\left(\frac{n}{3}\right) + \frac{1}{820}\tau_{8,3}(n) + \frac{4}{205}\tau_{8,3}\left(\frac{n}{2}\right) + \frac{1}{480}\tau_{8,6}(n).
\end{aligned}$$

As mentioned earlier, $\tau_{k,N}(n)$ denotes the n -th Fourier coefficient of the normalised newform of weight k and level N .

Next, we shall give the triple convolution sums required for the purpose of getting formulas for the representation numbers. All these are evaluated using our method and by using explicit basis for the space of modular forms of weight 12 and level 6, which is given in Table 2.

$$\begin{aligned}
W_{1,2,3}^{3,3,3}(n) &= \frac{1}{57600}(\sigma_3(n) + \sigma_3\left(\frac{n}{2}\right) + \sigma_3\left(\frac{n}{3}\right)) - \frac{1}{401472}\sigma_7(n) - \frac{83}{2509200}\sigma_7\left(\frac{n}{2}\right) - \frac{81}{2230400}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{9}{278800}\sigma_7\left(\frac{n}{6}\right) + \frac{697}{132201014400}\sigma_{11}(n) + \frac{82}{1032820425}\sigma_{11}\left(\frac{n}{2}\right) + \frac{12393}{29378003200}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{729}{114757825}\sigma_{11}\left(\frac{n}{6}\right) - \frac{37}{1958400}\tau_{8,2}(n) - \frac{63}{217600}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{13}{590400}\tau_{8,3}(n) - \frac{1}{12300}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad + \frac{1}{115200}\tau_{8,6}(n) - \frac{2429}{1014940800}\tau(n) + \frac{7112}{7929225}\tau\left(\frac{n}{2}\right) + \frac{2385531}{75180800}\tau\left(\frac{n}{3}\right) + \frac{209466}{293675}\tau\left(\frac{n}{6}\right) \\
&\quad + \frac{215}{6909888}\tau_{12,3}(n) - \frac{104}{107967}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{11}{313200}\tau_{12,6;1}(n) - \frac{1}{179712}\tau_{12,6;2}(n) - \frac{11}{268800}\tau_{12,6;3}(n),
\end{aligned}$$

$$\begin{aligned}
W_{2,2,3}^{3,3,3}(n) &= \frac{1}{28800}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{57600}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{20073600}\sigma_7(n) - \frac{713}{20073600}\sigma_7\left(\frac{n}{2}\right) - \frac{9}{2230400}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{9}{139400}\sigma_7\left(\frac{n}{6}\right) + \frac{41}{132201014400}\sigma_{11}(n) + \frac{697}{8262563400}\sigma_{11}\left(\frac{n}{2}\right) + \frac{729}{29378003200}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{12393}{1836125200}\sigma_{11}\left(\frac{n}{6}\right) - \frac{7}{979200}\tau_{8,2}(n) - \frac{63}{108800}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{1}{98400}\tau_{8,3}(n) - \frac{1}{6150}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad + \frac{1}{57600}\tau_{8,6}(n) - \frac{3059}{507470400}\tau(n) - \frac{12103}{63433800}\tau\left(\frac{n}{2}\right) + \frac{1112211}{37590400}\tau\left(\frac{n}{3}\right) + \frac{4400487}{4698800}\tau\left(\frac{n}{6}\right) \\
&\quad + \frac{133}{3454944}\tau_{12,3}(n) - \frac{367}{215934}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{7}{313200}\tau_{12,6;1}(n) - \frac{7}{359424}\tau_{12,6;2}(n) - \frac{19}{537600}\tau_{12,6;3}(n), \\
W_{2,3,3}^{3,3,3}(n) &= \frac{1}{57600}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{28800}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{20073600}\sigma_7(n) - \frac{1}{1254600}\sigma_7\left(\frac{n}{2}\right) - \frac{389}{10036800}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{9}{139400}\sigma_7\left(\frac{n}{6}\right) + \frac{17}{264402028800}\sigma_{11}(n) + \frac{1}{1032820425}\sigma_{11}\left(\frac{n}{2}\right) + \frac{6273}{14689001600}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{738}{114757825}\sigma_{11}\left(\frac{n}{6}\right) - \frac{7}{979200}\tau_{8,2}(n) - \frac{63}{108800}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{1}{98400}\tau_{8,3}(n) - \frac{1}{6150}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad + \frac{1}{57600}\tau_{8,6}(n) + \frac{18109}{2029881600}\tau(n) + \frac{9154}{7929225}\tau\left(\frac{n}{2}\right) - \frac{36309}{37590400}\tau\left(\frac{n}{3}\right) - \frac{36708}{293675}\tau\left(\frac{n}{6}\right) \\
&\quad - \frac{367}{31094496}\tau_{12,3}(n) + \frac{1064}{971703}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{7}{704700}\tau_{12,6;1}(n) + \frac{7}{808704}\tau_{12,6;2}(n) - \frac{19}{1209600}\tau_{12,6;3}(n), \\
W_{1,1,6}^{3,3,3}(n) &= \frac{1}{28800}\sigma_3(n) + \frac{1}{57600}\sigma_3\left(\frac{n}{6}\right) - \frac{349}{10036800}\sigma_7(n) - \frac{1}{1254600}\sigma_7\left(\frac{n}{2}\right) - \frac{9}{2230400}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{9}{139400}\sigma_7\left(\frac{n}{6}\right) + \frac{697}{132201014400}\sigma_{11}(n) + \frac{82}{1032820425}\sigma_{11}\left(\frac{n}{2}\right) + \frac{12393}{29378003200}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{729}{114757825}\sigma_{11}\left(\frac{n}{6}\right) - \frac{7}{979200}\tau_{8,2}(n) - \frac{63}{108800}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{1}{98400}\tau_{8,3}(n) - \frac{1}{6150}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad - \frac{1}{57600}\tau_{8,6}(n) - \frac{12103}{1014940800}\tau(n) - \frac{12236}{7929225}\tau\left(\frac{n}{2}\right) + \frac{4400487}{75180800}\tau\left(\frac{n}{3}\right) + \frac{2224422}{293675}\tau\left(\frac{n}{6}\right) \\
&\quad - \frac{367}{3454944}\tau_{12,3}(n) + \frac{1064}{107967}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{7}{78300}\tau_{12,6;1}(n) - \frac{7}{89856}\tau_{12,6;2}(n) + \frac{19}{134400}\tau_{12,6;3}(n), \\
W_{1,2,6}^{3,3,3}(n) &= \frac{1}{57600}(\sigma_3(n) + \sigma_3\left(\frac{n}{2}\right) + \sigma_3\left(\frac{n}{6}\right)) - \frac{83}{40147200}\sigma_7(n) - \frac{269}{8029440}\sigma_7\left(\frac{n}{2}\right) - \frac{9}{4460800}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{297}{4460800}\sigma_7\left(\frac{n}{6}\right) + \frac{41}{132201014400}\sigma_{11}(n) + \frac{697}{8262563400}\sigma_{11}\left(\frac{n}{2}\right) + \frac{729}{29378003200}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{12393}{1836125200}\sigma_{11}\left(\frac{n}{6}\right) - \frac{37}{1958400}\tau_{8,2}(n) - \frac{63}{217600}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{1}{196800}\tau_{8,3}(n) - \frac{29}{295200}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad - \frac{1}{115200}\tau_{8,6}(n) + \frac{889}{253735200}\tau(n) - \frac{2429}{63433800}\tau\left(\frac{n}{2}\right) + \frac{104733}{37590400}\tau\left(\frac{n}{3}\right) + \frac{2385531}{4698800}\tau\left(\frac{n}{6}\right) \\
&\quad - \frac{13}{3454944}\tau_{12,3}(n) + \frac{215}{431868}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{11}{1252800}\tau_{12,6;1}(n) - \frac{1}{718848}\tau_{12,6;2}(n) + \frac{11}{1075200}\tau_{12,6;3}(n),
\end{aligned}$$

$$\begin{aligned}
W_{1,3,6}^{3,3,3}(n) &= \frac{1}{57600}(\sigma_3(n) + \sigma_3\left(\frac{n}{3}\right) + \sigma_3\left(\frac{n}{6}\right)) - \frac{1}{2230400}\sigma_7(n) - \frac{1}{2509200}\sigma_7\left(\frac{n}{2}\right) - \frac{77}{2007360}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{163}{2509200}\sigma_7\left(\frac{n}{6}\right) + \frac{17}{264402028800}\sigma_{11}(n) + \frac{1}{1032820425}\sigma_{11}\left(\frac{n}{2}\right) + \frac{6273}{14689001600}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{738}{114757825}\sigma_{11}\left(\frac{n}{6}\right) - \frac{7}{1958400}\tau_{8,2}(n) - \frac{199}{652800}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{13}{590400}\tau_{8,3}(n) - \frac{1}{12300}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad - \frac{1}{115200}\tau_{8,6}(n) + \frac{9817}{2029881600}\tau(n) + \frac{862}{7929225}\tau\left(\frac{n}{2}\right) - \frac{7287}{37590400}\tau\left(\frac{n}{3}\right) + \frac{21336}{293675}\tau\left(\frac{n}{6}\right) \\
&\quad + \frac{215}{62188992}\tau_{12,3}(n) - \frac{104}{971703}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{11}{2818800}\tau_{12,6;1}(n) + \frac{1}{1617408}\tau_{12,6;2}(n) + \frac{11}{2419200}\tau_{12,6;3}(n), \\
W_{1,6,6}^{3,3,3}(n) &= \frac{1}{57600}\sigma_3(n) + \frac{1}{28800}\sigma_3\left(\frac{n}{6}\right) - \frac{1}{20073600}\sigma_7(n) - \frac{1}{1254600}\sigma_7\left(\frac{n}{2}\right) - \frac{9}{2230400}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{1993}{20073600}\sigma_7\left(\frac{n}{6}\right) + \frac{1}{264402028800}\sigma_{11}(n) + \frac{17}{16525126800}\sigma_{11}\left(\frac{n}{2}\right) + \frac{369}{14689001600}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{6273}{918062600}\sigma_{11}\left(\frac{n}{6}\right) - \frac{7}{979200}\tau_{8,2}(n) - \frac{63}{108800}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{1}{98400}\tau_{8,3}(n) - \frac{1}{6150}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad - \frac{1}{57600}\tau_{8,6}(n) + \frac{4577}{1014940800}\tau(n) + \frac{18109}{126867600}\tau\left(\frac{n}{2}\right) - \frac{9177}{18795200}\tau\left(\frac{n}{3}\right) - \frac{36309}{2349400}\tau\left(\frac{n}{6}\right) \\
&\quad + \frac{133}{31094496}\tau_{12,3}(n) - \frac{367}{1943406}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{7}{2818800}\tau_{12,6;1}(n) + \frac{7}{3234816}\tau_{12,6;2}(n) + \frac{19}{4838400}\tau_{12,6;3}(n), \\
W_{2,3,6}^{3,3,3}(n) &= \frac{1}{57600}(\sigma_3\left(\frac{n}{2}\right) + \sigma_3\left(\frac{n}{3}\right) + \sigma_3\left(\frac{n}{6}\right)) - \frac{1}{40147200}\sigma_7(n) - \frac{11}{13382400}\sigma_7\left(\frac{n}{2}\right) - \frac{163}{40147200}\sigma_7\left(\frac{n}{3}\right) \\
&\quad - \frac{797}{8029440}\sigma_7\left(\frac{n}{6}\right) + \frac{1}{264402028800}\sigma_{11}(n) + \frac{17}{16525126800}\sigma_{11}\left(\frac{n}{2}\right) + \frac{369}{14689001600}\sigma_{11}\left(\frac{n}{3}\right) \\
&\quad + \frac{6273}{918062600}\sigma_{11}\left(\frac{n}{6}\right) - \frac{7}{1958400}\tau_{8,2}(n) - \frac{199}{652800}\tau_{8,2}\left(\frac{n}{3}\right) - \frac{1}{196800}\tau_{8,3}(n) - \frac{29}{295200}\tau_{8,3}\left(\frac{n}{2}\right) \\
&\quad + \frac{1}{115200}\tau_{8,6}(n) + \frac{431}{1014940800}\tau(n) + \frac{9817}{126867600}\tau\left(\frac{n}{2}\right) + \frac{2667}{9397600}\tau\left(\frac{n}{3}\right) - \frac{7287}{2349400}\tau\left(\frac{n}{6}\right) \\
&\quad - \frac{13}{31094496}\tau_{12,3}(n) + \frac{215}{3886812}\tau_{12,3}\left(\frac{n}{2}\right) + \frac{11}{11275200}\tau_{12,6;1}(n) + \frac{1}{6469632}\tau_{12,6;2}(n) - \frac{11}{9676800}\tau_{12,6;3}(n).
\end{aligned}$$

We are now ready to derive the required formulas for the number of representations. Let $r_{Q_i}(n)$ denote the number of representations of a natural number n by the quadratic form Q_i , $i = 1, 2$ as defined above. Then, we have

$$\begin{aligned}
r_{Q_1}(n) &= \sum_{\substack{a,b,c \in \mathbb{N} \cup \{0\} \\ a+b+2c=n}} \left(\sum_{\substack{(x_1, \dots, x_8) \in \mathbb{Z}^8 \\ Q_8(x_1, \dots, x_8)=a}} 1 \right) \left(\sum_{\substack{(x_9, \dots, x_{16}) \in \mathbb{Z}^8 \\ Q_8(x_9, \dots, x_{16})=b}} 1 \right) \left(\sum_{\substack{(x_{17}, \dots, x_{24}) \in \mathbb{Z}^8 \\ Q_8(x_{17}, \dots, x_{24})=c}} 1 \right) \\
&= 2s_8(n) + s_8(n/2) + \sum_{\substack{a,b \in \mathbb{N} \\ a+b=n}} s_8(a)s_8(b) + 2 \sum_{\substack{a,b \in \mathbb{N} \\ a+2b=n}} s_8(a)s_8(b) + \sum_{\substack{a,b,c \in \mathbb{N} \\ a+b+2c=n}} s_8(a)s_8(b)s_8(c)
\end{aligned}$$

$$\begin{aligned}
&= 48\sigma_3(n) + 24\sigma_3(n/2) + 432\sigma_3(n/3) + 216\sigma_3(n/6) + 24^2 W_{1,1}^{3,3}(n) + 2 \times 24^2 W_{1,2}^{3,3}(n) \\
&\quad + 48 \times 216 W_{1,3}^{3,3}(n) + 48 \times 216 W_{1,6}^{3,3}(n) + 48 \times 216 W_{2,3}^{3,3}(n) + 216^2 W_{1,1}^{3,3}(n/3) \\
&\quad + 2 \times 216^2 W_{1,2}^{3,3}(n/3) + 24^3 W_{1,1,2}^{3,3,3}(n) + 24^2 \times 216 W_{1,1,6}^{3,3,3}(n) + 24^2 \times 432 W_{1,2,3}^{3,3,3}(n) \\
&\quad + 24 \times 216^2 W_{2,3,3}^{3,3,3}(n) + 48 \times 216^2 W_{1,3,6}^{3,3,3}(n) + 216^3 W_{1,1,2}^{3,3,3}(n/3).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
r_{Q_2}(n) &= \sum_{\substack{a,b,c \in \mathbb{N} \cup \{0\} \\ a+2b+2c=n}} \left(\sum_{\substack{(x_1, \dots, x_8) \in \mathbb{Z}^8 \\ Q_8(x_1, \dots, x_8)=a}} 1 \right) \left(\sum_{\substack{(x_9, \dots, x_{16}) \in \mathbb{Z}^8 \\ Q_8(x_9, \dots, x_{16})=b}} 1 \right) \left(\sum_{\substack{(x_{17}, \dots, x_{24}) \in \mathbb{Z}^8 \\ Q_8(x_{17}, \dots, x_{24})=c}} 1 \right) \\
&= s_8(n) + 2s_8(n/2) + \sum_{\substack{a,b \in \mathbb{N} \\ a+b=n/2}} s_8(a)s_8(b) + 2 \sum_{\substack{a,b \in \mathbb{N} \\ a+2b=n}} s_8(a)s_8(b) + \sum_{\substack{a,b,c \in \mathbb{N} \\ a+2b+2c=n}} s_8(a)s_8(b)s_8(c) \\
&= 24\sigma_3(n) + 48\sigma_3(n/2) + 216\sigma_3(n/3) + 432\sigma_3(n/6) + 2 \times 24^2 W_{1,2}^{3,3}(n) + 48 \times 216 W_{1,6}^{3,3}(n) \\
&\quad + 48 \times 216 W_{2,3}^{3,3}(n) + 2 \times 216^2 W_{1,2}^{3,3}(n/3) + 48 \times 216 W_{1,3}^{3,3}(n/2) + 24^2 W_{1,1}^{3,3}(n/2) \\
&\quad + 216^2 W_{1,1}^{3,3}(n/6) + 24^3 W_{1,2,2}^{3,3,3}(n) + 24^2 \times 432 W_{1,2,6}^{3,3,3}(n) + 24 \times 216^2 W_{1,6,6}^{3,3,3}(n) \\
&\quad + 24^2 \times 216 W_{2,2,3}^{3,3,3}(n) + 48 \times 216^2 W_{2,3,6}^{3,3,3}(n) + 216^3 W_{1,2,2}^{3,3,3}(n/3).
\end{aligned}$$

Now, using the double and triple convolution sums presented in this section and the convolution sums $W_{1,1,2}^{3,3,3}(n)$, $W_{1,2,2}^{3,3,3}(n)$ presented in Section 4, we derive the following explicit formulas:

Theorem 5.1. *Let n be a natural number. Then the number of representations of n by the quadratic forms Q_1 and Q_2 are given as follows:*

$$\begin{aligned}
r_{Q_1}(n) &= \frac{408}{50443} \sigma_{11}(n) + \frac{6144}{50443} \sigma_{11}\left(\frac{n}{2}\right) + \frac{297432}{50443} \sigma_{11}\left(\frac{n}{3}\right) + \frac{4478976}{50443} \sigma_{11}\left(\frac{n}{6}\right) + \frac{225720}{11747} \tau(n) \\
&\quad + \frac{19132416}{11747} \tau\left(\frac{n}{2}\right) + \frac{164549880}{11747} \tau\left(\frac{n}{3}\right) + \frac{13947531264}{11747} \tau\left(\frac{n}{6}\right) - \frac{393984}{35989} \tau_{12,3}(n) \\
&\quad + \frac{71000064}{35989} \tau_{12,3}\left(\frac{n}{2}\right) + \frac{1152}{29} \tau_{12,6;1}(n),
\end{aligned} \tag{19}$$

$$\begin{aligned}
r_{Q_2}(n) &= \frac{24}{50443} \sigma_{11}(n) + \frac{6528}{50443} \sigma_{11}\left(\frac{n}{2}\right) + \frac{17496}{50443} \sigma_{11}\left(\frac{n}{3}\right) + \frac{4758912}{50443} \sigma_{11}\left(\frac{n}{6}\right) + \frac{74736}{11747} \tau(n) \\
&\quad + \frac{3611520}{11747} \tau\left(\frac{n}{2}\right) + \frac{54482544}{11747} \tau\left(\frac{n}{3}\right) + \frac{2632798080}{11747} \tau\left(\frac{n}{6}\right) + \frac{277344}{35989} \tau_{12,3}(n) \\
&\quad - \frac{6303744}{35989} \tau_{12,3}\left(\frac{n}{2}\right) + \frac{288}{29} \tau_{12,6;1}(n).
\end{aligned} \tag{20}$$

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APPENDIX

In this section, we give explicit expressions for the newforms listed in Tables 1 and 2. As mentioned earlier, $E_k(z)$ denotes the normalised Eisenstein series of weight k for the full modular group, $\Delta(z)$ denotes the Ramanujan delta function of weight 12 and $\Delta_{k,N;j}(z)$ denotes the j -th normalised newform of weight k , level N . (In the case when there is only one normalised newform of a given weight and level, we drop the subscript j .) We also use the Dedekind eta function $\eta(z)$ in our expressions, which is defined by $\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$, where $q = e^{2\pi i z}$.

$$\begin{aligned}
\Delta_{6,3}(z) &= \eta^6(z)\eta^6(3z), \\
\Delta_{6,6}(z) &= \frac{-7}{480} \left\{ \frac{50}{1911} E_6(z) + \frac{160}{1911} E_6(2z) + \frac{135}{637} E_6(3z) + \frac{432}{637} E_6(6z) + \frac{840}{13} \Delta_{6,3}(z) \right. \\
&\quad \left. + \frac{7680}{13} \Delta_{6,3}(2z) + \frac{1}{2} D E_4(z) - E_2(6z) E_4(z) \right\}, \\
\Delta_{8,2}(z) &= \eta^8(z)\eta^8(2z), \\
\Delta_{8,3}(z) &= \eta^{12}(z)\eta^4(3z) + 81\eta^6(z)\eta^4(3z)\eta^6(9z) + 18\eta^9(z)\eta^4(3z)\eta^3(9z), \\
\Delta_{8,6}(z) &= \frac{1}{240} (E_4(z)E_4(6z) - E_4(2z)E_4(3z)), \\
\Delta_{10,2}(z) &= \frac{31}{63} \frac{\eta^{16}(2z)}{\eta^8(z)} E_6(z) - \frac{4}{2079} E_{10}(z) + \frac{4}{2079} E_{10}(2z), \\
\Delta_{10,3;1}(z) &= -\frac{45}{17248} E_{10}(z) + \frac{45}{17248} E_{10}(3z) + \frac{3355}{240 \times 5292} (E_4(z) - E_4(3z)) E_6(z) \\
&\quad - \frac{61}{189} \Delta_{6,3}(z) E_4(z), \\
\Delta_{10,3;2}(z) &= -\frac{9}{4312} E_{10}(z) + \frac{9}{4312} E_{10}(3z) + \frac{671}{240 \times 1323} (E_4(z) - E_4(3z)) E_6(z) \\
&\quad - \frac{11}{189} \Delta_{6,3}(z) E_4(z), \\
\Delta_{10,6}(z) &= -\frac{143}{243} \Delta_{10,2}(z) - 297 \Delta_{10,2}(3z) + \frac{11}{9} \Delta_{10,3;1}(z) + \frac{1408}{27} \Delta_{10,3;1}(2z) + \frac{7}{18} \Delta_{10,3;2}(z) \\
&\quad - \frac{896}{27} \Delta_{10,3;3}(2z) - \frac{11}{486} \Delta_{4,6}(z) E_6(z), \\
\Delta_{12,3}(z) &= \frac{98}{81} \Delta(z) - \frac{275562}{81} \Delta(3z) - \frac{17}{81} \Delta_{6,3}(z) E_6(z),
\end{aligned}$$

$$\begin{aligned}
\Delta_{12,6;1}(z) &= -\frac{203}{323}\Delta(z) - \frac{928}{323}\Delta(2z) - \frac{147987}{323}\Delta(3z) - \frac{676512}{323}\Delta(6z) - \frac{2727}{2261}\Delta_{12,3}(z) \\
&\quad + \frac{21600}{323}\Delta_{12,3}(2z) + \frac{783}{266}\Delta_{6,3}(z)E_6(6z) - \frac{29}{266}\Delta_{6,3}(z)E_6(2z), \\
\Delta_{12,6;2}(z) &= -\frac{16051}{16150}\Delta(z) - \frac{153728}{8075}\Delta(2z) + \frac{4781511}{16150}\Delta(3z) + \frac{340028928}{8075}\Delta(6z) - \frac{168021}{129200}\Delta_{12,3}(z) \\
&\quad + \frac{27648}{323}\Delta_{12,3}(2z) + \frac{10989}{3325}\Delta_{6,3}(z)E_6(6z) - \frac{27}{3325}\Delta_{6,3}(z)E_6(2z) - \frac{1}{400}\Delta_{4,6}(z)E_8(z), \\
\Delta_{12,6;3}(z) &= -\frac{679}{5814}\Delta(z) + \frac{112384}{8721}\Delta(2z) + \frac{311283}{646}\Delta(3z) - \frac{11477376}{323}\Delta(6z) + \frac{6223}{5168}\Delta_{12,3}(z) \\
&\quad - \frac{25600}{323}\Delta_{12,3}(2z) - \frac{27}{133}\Delta_{6,3}(z)E_6(6z) + \frac{407}{3591}\Delta_{6,3}(z)E_6(2z) + \frac{1}{432}\Delta_{4,6}(z)E_8(z).
\end{aligned}$$

REFERENCES

- [1] A. Alaca, S. Alaca and E. Ntienjem, *The convolution sum $\sum_{al+bm=n} \sigma(l)\sigma(m)$ for $(a,b) = (1,28), (4,7), (1,14), (2,7), (1,7)$* , Kyungpook Math. J. **59** (2019), no. 3, 377–389.
- [2] A. Alaca, S. Alaca and K. S. Williams, *Evaluation of the convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$* , Adv. Theor. Appl. Math. **1** (2006), no. 1, 27–48.
- [3] A. Alaca, S. Alaca and K. S. Williams, *Evaluation of the convolution sums $\sum_{l+18m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+9m=n} \sigma(l)\sigma(m)$* , Int. Math. Forum **2** (2007), no. 1-4, 45–68.
- [4] A. Alaca, S. Alaca and K. S. Williams, *Evaluation of the convolution sums $\sum_{l+24m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+8m=n} \sigma(l)\sigma(m)$* , Math. J. Okayama Univ. **49** (2007), 93–111.
- [5] A. Alaca, S. Alaca and K. S. Williams, *The convolution sum $\sum_{m< n/16} \sigma(m)\sigma(n-16m)$* , Canad. Math. Bull. **51** (2008), no. 1, 3–14.
- [6] S. Alaca and Y. Kesencioglu, *Evaluation of the convolution sums $\sum_{l+27m=n} \sigma(l)\sigma(m)$ and $\sum_{l+32m=n} \sigma(l)\sigma(m)$* , J. Number Theory **1** (2016), 1–13.
- [7] S. Alaca, F. Uygun and K. S. Williams, *Some arithmetic identities involving divisor functions*, Funct. Approx. Comment. Math. **46** (2012), no. 2, 261–271.
- [8] S. Alaca and K. S. Williams, *Evaluation of the convolution sums $\sum_{l+6m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+3m=n} \sigma(l)\sigma(m)$* , J. Number Theory **124** (2007), no. 2, 491–510.
- [9] Z.S. Aygin, *Eisenstein series and convolution sums*, Ramanujan J. **48**(3) (2019) 495–508.
- [10] M. Besge: *Extrait d'une lettre de M. Besge à M. Liouville*, J. Math. Pures Appl. **7**, 256 (1862).
- [11] H.H. Chan and S. Cooper, *Powers of theta functions*, Pac. J. Math. **235** (2008), 1–14.
- [12] N. Cheng and K. S. Williams, *Evaluation of some convolution sums involving the sum of divisors functions*, Yokohama Math. J. **52** (2005), no. 1, 39–57.
- [13] B. Cho, *Convolution sums of a divisor function for prime levels*, Int. J. Number Theory, **16** (2020), 537–546.
- [14] S. Cooper and P.C. Toh, *Quintic and septic Eisenstein series*, Ramanujan J. **19** (2009), 163–181.
- [15] S. Cooper and D. Ye, *Evaluation of the convolution sums $\sum_{l+20m=n} \sigma(l)\sigma(m)$, $\sum_{4l+5m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+5m=n} \sigma(l)\sigma(m)$* , Int. J. Number Theory, **6**, (2014) 1385–1394.
- [16] J.W.L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14**, (1885) 156–163.

- [17] J. G. Huard, Z. M. Ou, B. K. Spearman and K. S. Williams, *Elementary evaluation of certain convolution sums involving divisor functions*, in Number Theory for the Millennium, II (Urbana, IL, 2000) (A. K. Peters, Natick, MA, 2002), 229–274.
- [18] B. Köklüce, *Evaluation of four convolution sums and representation of integers by certain quadratic forms in twelve variables*, J. Math. Comput. Sci. **12** (2022), Article ID 52, DOI : 10.28919/jmcs/7016.
- [19] D. B. Lahiri, *On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$ -I*, Bull. Calcutta Math. Soc. **38** (1946), 193–206.
- [20] D. B. Lahiri, *On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$, Part II*, Bull. Calcutta Math. Soc. **39** (1947), 33–52.
- [21] M. Lemire and K.S. Williams, *Evaluation of two convolution sums involving the sum of divisor functions*, Bull. Aust. Math. Soc. **73** (2005), 107–115.
- [22] LMFDB, The database of L-functions, modular forms, and related objects, <http://www.lmfdb.org/>
- [23] G. A. Lomadze, *Representation of numbers by sums of the quadratic forms $x_1^2 + x_1x_2 + x_2^2$* , Acta Arith., **54** (1989), 9–36 (in Russian).
- [24] G. Melfi, *Some problems in elementary number theory and modular forms*, Ph.D.Thesis, University of Pisa,1997.
- [25] G. Melfi, *On some modular identities*, in Number Theory, Diophantine equations, Computational and Algebraic Aspects: Proceedings of the International Conference held in Eger, Hungary. Walter de Grutyer and Co. (1998), 371–382.
- [26] E. Ntienjem, *Evaluation of convolution sums involving the sum of divisors function for levels 48 and 64*, Integers **17** (2017), Paper No. A49, 22 pp.
- [27] E. Ntienjem, *Evaluation of the convolution sum involving the sum of divisors function for 22, 44 and 52*, Open Math. **15** (2017), no. 1, 446–458.
- [28] E. Ntienjem, *Elementary evaluation of convolution sums involving the divisor function for a class of levels*, North-West. Eur. J. Math. **5** (2019), 101–165.
- [29] E. Ntienjem, *Evaluation of convolution sums entailing mixed divisor functions for a class of level*, New Zealand J. Math. **50** (2020), 125–180.
- [30] Yoon Kyung Park, *Evaluation of the convolution sums $\sum_{ak+bl+cm=n} \sigma(k)\sigma(l)\sigma(m)$ with $\text{lcm}(a, b, c) \leq 6$* , J. Number Theory **168** (2016), 257–275.
- [31] Yoon Kyung Park, *Evaluation of the convolution sums $\sum_{ak+bl+cm=n} \sigma(k)\sigma(l)\sigma(m)$ with $\text{lcm}(a, b, c) = 7, 8$ or 9*, Int. J. Number Theory **14**(6) (2018), 1637–1650.
- [32] B. Ramakrishnan and Brundaban Sahu, *Evaluation of the convolution sums $\sum_{l+15m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+5m=n} \sigma(l)\sigma(m)$ and an application*, Int. J. Number Theory **9**, (2013) 799–809.
- [33] B. Ramakrishnan and Brundaban Sahu, *Evaluation of convolution sums and some remarks on cusp forms of weight 4 and level 12*, Math. J Okayama Univ. **59** (2017), [2016 on cover], 71–79.
- [34] B. Ramakrishnan and Brundaban Sahu, *On the number of representations of certain quadratic forms of sixteen variables*, Int. J. Number Theory **10** (2014), 1929 – 1937.
- [35] B. Ramakrishnan and Brundaban Sahu, *On the number of representations of certain quadratic forms in 20 and 24 variables*, Functiones et Approximatio **54** (2016), no. 2, 151–161.
- [36] B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, *Representations of an integer by some quaternary and octonary quadratic forms*, Geometry, Algebra, Number Theory and their Information Technology Applications (GANITA), Springer Proc. Math. and Stat. **251** (2018), 383 – 409.
- [37] B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, *On the number of representations of certain quadratic forms in 8 variables*, Automorphic Forms and Related Topics, Samuele Anni et al. (eds.), 215–224, Contemp. Math. **732**, Amer. Math. Soc. Providence, RI, 2019.
- [38] B. Ramakrishnan, Brundaban Sahu and Anup Kumar Singh, *A simple extension of Ramanujan-Serre derivative map and some applications*, Ramanujan J. **61** (2023), 1379–1410.
- [39] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916) 159–184.

- [40] E. Royer, *Evaluating convolution sums of the divisor function by quasimodular forms*, Int. J. Number Theory **3** (2007), no. 2, 231–261.
- [41] SageMath, The Sage Mathematics Software System (Version 6.9), The Sage Developers, 2021, <https://www.sagemath.org>
- [42] X.L. Tian, O.X.M. Yao and E.X.W.Xia, *Evaluation of the convolution sum $\sum_{l+25m=n} \sigma(l)\sigma(m)$* , Int. J. Number Theory **10**, (2014) 1421-1430.
- [43] K. S. Williams, *The convolution sum $\sum_{m < n/9} \sigma(m)\sigma(n - 9m)$* , Int. J. Number Theory **1** (2005), no. 2, 193–205.
- [44] K. S. Williams, *The convolution sum $\sum_{m < n/8} \sigma(m)\sigma(n - 8m)$* , Pacific J. Math. **228** (2006), no. 2, 387–396.
- [45] K. S. Williams, Number Theory in the spirit of Liouville, *London Mathematical Student Texts* **76**, Cambridge Univ. Press, 2011.
- [46] O. X. M. Yao and E. X. W. Xia *Evaluation of the convolution sum $\sum_{i+3j=n} \sigma(m)\sigma_3(j)$ and $\sum_{3i+j=n} \sigma(m)\sigma_3(j)$* , Int. J. Number Theory **10** (2014), No. 1, 115–123.

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