Mathematical Sciences@NISER

Discussion Meeting on Basic Sciences
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National Institute of Science Education and Research, Bhubaneswar

Academic Programs

Academic Programs

- Integrated M.Sc in Mathematics (major as well as minor)

  Core Courses: Real Analysis, Group Theory, Discrete Mathematics, Metric spaces, Linear Algebra, Probability Theory, Lebesgue Integration, Rings and Modules, Differential Equations, Number Theory, Calculus of several variables, Field Theory, Complex Analysis, Graph Theory, Functional Analysis, Representations of Finite groups, Topology, Geometry of curves and surfaces.

  Elective Courses: Partial Differential Equations, Algebraic Geometry, Operator Theory, Operator Algebras, Algebraic Topology, Introduction to Manifolds, Lie group and Lie Algebra, Algebraic Number Theory, Classical Groups, Theory of Computation, Algorithm, ...

Academic Programs

Academic Programs

- Ph.D in Mathematics

  Core Courses: Algebra I, Algebra II, Analysis I, Analysis II, Topology I, Topology II, Complex Analysis.

- Integrated M.Sc-Ph.D Program in Mathematics (proposal submitted, awaiting for HBNI approval)

Research/ Outreach Activities

Research/ Outreach Activities

- Organise Workshops/ Lecture series on various topics of current research.
- Visitor program for research collaborations.
- We invite (around 30-40) students every summer through Indian Academy of Sciences/ INSPIRE Scholarship or through NISER summer fellowship to spend *two months* at NISER to work with a faculty member for a project.
- Summer Outreach Program in Mathematics (started from 2016 onwards): It is a two weeks program covering some topics of master level courses meant for students having less exposure.

  The aim of this program is to give the students exposure to Mathematics through regular lectures/ tutorials/ interactions and to motivate students to pursue Mathematics as a research career.

Facilities and Infrastructure

Facilities and Infrastructure

- Central Library facilities:
  (with 6575 printed mathematics books and 2281 e-books
  390 subscribed Mathematics and computer science Journals).
  Databases like MathSciNet, JSTOR, Project Euclid, Science Direct
- Departmental Library (with 1277 printed mathematics books) with reading room facility.
- Computer Laboratory

Faculty members/ Research areas

Faculty members/ Research areas

- 20 Faculty members, 16 Ph.D students, 11 Postdoctoral fellows, 60 master students

  13 Ph.D completed

- Algebraic Geometry and Complex Geometry (Krishanu Dan, Ritwik Mukherjee, Chitrabhanu Chaudhuri)
- Analysis and Probability (Anil Karn, Dinesh Kumar Keshari, Sutanu Roy, Panchugopal Bikram, Ramesh Manna, Sanjay Parui, Nabin Jana)
- Discrete Mathematics and Cryptography (Binod Kumar Sahoo, Kamal Lochan Patra, Deepak Kumar Dalai, Kaushik Majumder)
- Partial Differential Equations (Anupam Pal Choudhury, Manas Ranjan Sahoo)
- Number Theory (Brundaban Sahu, Jaban Meher, K. Senthil Kumar, Sudhir Pujahari)

Algebraic Geometry

Algebraic Geometry

- Let ϕ be a differentiable function in real variable. Fix r-points $t_1, \cdots, t_r$ where ϕ is defined and integers $m_1, \cdots, m_r \geq 1$ such that $\sum_{i=1}^{r} m_i = d + 1$. Then there is a "unique" polynomial $f(x)$ of degree d (by the Fundamental Theorem of Algebra) such that $f^{(j-1)}(t_i) = \phi^{(j-1)}(t_i)$ for $1 \leq i \leq r$ and $1 \leq j \leq m_i$.
  In other words, we get a polynomial approximation of ϕ. As d increases, we get better approximation.
- Analogously, given a set of points $\{(t_i, s_i) : 1 \leq i \leq r\}$ and integers $m_1, \cdots, m_r \geq 1$, what is the least positive integer $d \geq 1$ such that there exists a polynomial $f(x, y)$ of degree d having (t_i, s_i) as roots of multiplicity m_i, for each $1 \leq i \leq r$?
- Working over $\mathbb{C}$, this problem has a beautiful geometric interpretation: find the least degree curve in $\mathbb{P}^2$ passes through the points $(t_i, s_i, 1)$ with multiplicity m_i.
- This classical problem was partially answered by M. Nagata (1959) in his work on Hilbert's 14-th problem and gave a conjectural lower bound, known as *Nagata Conjecture:*

Algebraic Geometry

Algebraic Geometry

- Nagata Conjecture: Given $r \geq 9$ points $P_1, \cdots, P_r \in \mathbb{P}^2$ in "very general position" and integers $m_1, \cdots, m_r \geq 1$, if a curve $C \subset \mathbb{P}^2$ of degree d passes through P_i with multiplicity m_i, then
$$d > \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i.$$
Nagata proved this result when r is a perfect square. Rest of the cases are still open.
- An equivalent formulation of Nagata Conjecture comes from *Seshadri Constants:* for r "very general" points in $\mathbb{P}^2$, $\varepsilon\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); r\big) = 1/\sqrt{r}$. Seshadri constants were defined by J.P. Demailly (1992), in honour of C. S. Seshadri. Effective bounds of Seshadri constants can be applied to determine global generation of adjoint line bundles.
- Krishanu and his collaborators (2022, work on progress) derived effective bounds of Seshadri constants on higher dimensional varieties.
- Member involved: Dr. Krishanu Dan (krishanu@niser.ac.in)

Analysis and Probability

Analysis and Probability

- The group works on local smoothing of Fourier integral operators with application to the wave equation and maximal operators. In particular, Fourier restriction problems for curves in $\mathbb{R}^d$.
- Fourier restriction theory plays a key role in quantum mechanics. In fact, this theory is used to estimate the size of the solutions of the Schrödinger equation in terms of the size of the initial datum.
- More precisely, for a compactly supported bounded function $f : \mathbb{R}^d \to \mathbb{R}$ define the Fourier tranform:

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\xi}\, dx.$$

- For such a function f, the group is interested to study the behaviour of f when restricted onto a subset of $\mathbb{R}^d$. For a subset $S \subset \mathbb{R}^d$ one asks if the following holds:

$$\|\hat{f}\|_{L^q(S)} \leq C_{p,q}\|f\|_{L^p(\mathbb{R}^d)} \textit{ for some constant } C_{p,q}.$$

This has been one of the fundamental problems in analysis since 1970's.

Analysis and Probability

Analysis and Probability

- For S is paraboloid or sphere in $\mathbb{R}^{d+1}$, this problems have attracted a lot of attention by many mathematicians.
- An interesting result (Bourgain (1998),..., Stovall(2019)) is that extremizers $\left(\frac{\|\hat{f}\|_{L^q(S)}}{\|f\|_{L^p}} = \sup_g \frac{\|\hat{g}\|_{L^q(S)}}{\|g\|_{L^p}}\right)$ for restriction to paraboloid exist in all dimensions d and for all restriction pairs (p, q).
- Interested similar questions for restriction to polynomial curves in $\mathbb{R}^d$.
- Ramesh and collaborators (2022) derived the existence of extremizers for monomial curves in $\mathbb{R}^d$.
- The group also interested to study time-frequency analysis and nonlinear dispersive equations; the interplay between these with harmonic analysis.
- Member involved: Dr. Ramesh Manna (rameshmanna@niser.ac.in)

Analysis and Probability

Analysis and Probability

- The notion of *symmetry* plays a fundamental role in Biology (e.g. bilateral symmetry), Chemistry (e.g. molecular symmetry), Mathematics (e.g. isometry), Physics (e.g. gauge symmetry).

- For example:
  • Distance preserving transformations or isometries of the three-dimensional space are combinations of translations and rotations.
  • Symmetries of a finite set are given by the permutations of the elements of the set;

- In particular, the notion of the isometry group of Riemannian manifolds has essential applications in Mathematics and Physics.

- How to generalise the notion of isometry for non-commutative spaces (e.g. matrix algebra, non-commutative 2-torus, complex quantum plane, quantum spheres,...)?

- Debashish Goswami (2009) developed the theory of *quantum isometry groups*, generalising the notion of isometry group for compact Riemannian manifolds.

Analysis and Probability

Analysis and Probability

- This theory was applied to compute the quantum isometry group of certain non-commutative (compact) space appearing in the standard model in particle physics [Chamseddine-Connes 2010].
- On the other hand, quantum field theory on non-commutative spaces is based on a deformation of Minkowski space to a non-commutative (non-compact) space M.
- Can we construct the symmetries of M by extending "Lorentz quantum group" (which is a suitable analogue of the homogeneous symmetries of M), by the "translation quantum group"?
- However, the "translation quantum group" is a *braided quantum group*.
- Sutanu and his collaborators (in 2021) constructed braided versions of quantum $E(2)$ group.
- Member involved: Dr. Sutanu Roy (sutanu@niser.ac.in)

Analysis and Probability

Analysis and Probability

- *Probability theory* is a branch of Mathematics which is closely related to scientific experiments and phenomenons. Hence application of it is wide spread and there are immense potential to have inter-disciplinary collaborations.
  The group is interested on the following:
- Analysing mean field models in the theoretical condensed matter physics.
- To analyse population Genetics and Evolution, theory of contagions, stochastic modelling of biological systems in the field of Biology.
- To analyse phase transitions, equilibrium dynamics, chemical analysis in the field of Chemistry.
- Economic modelling through probability and stochastic processes in the field of Economics.
- Nabin and his collaborator used the large scale deviation principle to analyse the phase transition in the mean-field models of statistical physics.
- Member involved: Dr. Nabin Kumar Jana (nabinjana@niser.ac.in)

Discrete Mathematics and Cryptography

Discrete Mathematics and Cryptography

- *Cryptography* is the practice and study of techniques for secure communication in the presence of third parties called adversaries.

- It is a branch of Mathematics and Computer science.

  Further, due to the recent development of quantum computers, Physics has great involvement in quantum cryptography.

- Its foundation is based on various concepts such as :
  - number theory, algebra, discrete mathematics, probability theory and statistics from Mathematics and
  - theory of computation and complexity, algorithm from Computer science.

Cryptography

Current research interests:

Cryptography

Current research interests:

- Analysis and construction of cryptographically significant Boolean functions.
- Analysis and design of stream ciphers.
- Analysis and design of post-quantum lattice based signature schemes

- Deepak and his collaborators (2021, 2022) studied cryptanalysis on Lizard and Grain-128a (which are two recent popular stream ciphers).

- Member involved: Dr. Deepak Kumar Dalai (deepak@niser.ac.in)

Partial Differential Equations

Partial Differential Equations

- The research group works in conservation laws in one space dimension as well as in multi-dimension and delta-waves applicable for conservation laws of the particular type, namely, non-strictly hyperbolic.
- The main goal of this group is to show the existence of solution in some suitable function space (functions of bounded variation).
- They study uniqueness of the solution by imposing physically proper conditions, known as entropy conditions.
- They also study the large time behaviour of the solutions and structure of the entropy admissible solution. Since these solutions are important equations coming from physics, only the existence may not very useful for practical purposes. Hence, this group is also interested to study numerical schemes for these system, their convergence analysis and order of convergence.

Partial Differential Equation

Partial Differential Equation

Important models under work on progress

- Large scale structure formation of the universe, Adhesion models
- Zero-pressure gas dynamics equations
- General one dimensional conservation laws applicable for many model
- Conservation laws with discontinuous flux [two phase flow in discontinuous porous medium, sedimentation procedure, traffic flow on highways with different flow density, blood flow, flow of gas in a non constant channel and fabrication of semiconductor device]

- Manas and his collaborators (2021) found explicit formula for the solution of one dimensional zero-pressure gas dynamics system for initial and boundary value problem (which is new for the boundary value problem).
- Members involved: Dr. Manas Ranjan Sahoo (manas@niser.ac.in), Dr. Anupam Pal Chowdhury (anupam@niser.ac.in)

Number Theory

Number Theory

- Modular forms and arithmetic of Fourier coefficients
- Differential operators on modular forms
- L-functions and its Zeros
- Transcendental Number Theory, Multiple Zeta values

Understanding Transcendental Numbers

Understanding Transcendental Numbers

- The complex numbers which are roots of some non-zero polynomial over rational numbers called algebraic numbers (examples: $\sqrt{2}, i$) . The complex numbers which are not algebraic are called transcendental numbers (examples: e, π).
- The famous Hilbert's 7th Problem asks if

$$2^{\sqrt{2}} = e^{\sqrt{2}\log 2}$$

  is transcendental. The answer is yes and was proved by Gelfond (1934) and Schneider (1935) independently.
- The famous Euler's identity

$$e^{\pi i} + 1 = 0$$

  motivates to study the linear independence and algebraic independence problems related to the values of e^z.
- To understand the algebraic nature of numbers like $\pi, e+\pi, e\pi, e^\pi, \pi^e$, etc., that are related to exponential function, one may need to understand the values of Weierstrass elliptic, zeta, and sigma functions.

Understanding Transcendental Numbers

Understanding Transcendental Numbers

- For example, Nesterenko's (1995) proof of the algebraic independence of π, e^{π} involves Weierstrass elliptic functions. On the other hand, it is well known that the Weierstrass's functions are very useful in evaluating elliptic integrals.
- One of the several themes of transcendental number theory is to find the transcendence degrees of the fields generated by the values of the classical exponential function and the Weierstrass's elliptic, zeta, and sigma functions over $\mathbb{Q}$.
- When these fields have transcendence degrees at least two over the field of rational numbers?
- This requires the study of algebraic subgroups of commutative algebraic groups connected with Weierstrass elliptic, zeta, and sigma functions.
- Recently, Senthil (2021) proved that the transcendence degrees of the fields generated by certain values of Weierstrass elliptic and zeta function is at least two over $\mathbb{Q}$.
- Member involved: Dr. K. Senthil Kumar (senthil@niser.ac.in)

Convolution Sums of divisor functions

Convolution Sums of divisor functions

- Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of positive integers.
- For $r, n \in \mathbb{N}$, let $\sigma_r(n) = \sum_{d|n, d\in\mathbb{N}} d^r$
- $\sigma_0(n)$ is the number of divisors of n.
- $\sigma_1(n)$ is the sum of divisors of n. (Denote by $\sigma(n)$)
- For $a, b, r, s, n \in \mathbb{N}$, consider the following sum:

$$W_{a,b}^{r,s}(n) := \sum_{\substack{l,m\in\mathbb{N} \\ al+bm=n}} \sigma_r(l)\sigma_s(m).$$

- For $r = s = 1$, we denote the above sum by $W_{a,b}(n) = \sum_{\substack{l,m\in\mathbb{N} \\ al+bm=n}} \sigma(l)\sigma(m)$.
- Early evaluation by Besge (1862), Glaisher (1885), Ramanujan (1916), Lahiri (1947)

Convolution Sums of divisor functions

Convolution Sums of divisor functions

- Several works on this direction from starting 1990..
- Elementary evaluation, (p, k) parametrization, modular forms and qusimodular forms.
- Interested to evaluate the convolution sum of divisor functions (may be used in various ways, like representation of natural number by certain quadratic forms).
- (Jacobi) The number of representations of a natural number n as the sum of four squares is $8\sigma(n) - 32\sigma(n/4)$.
- The number of representations of a natural number n as the sum of eight squares can be expressed in terms of $W_{1,1}(n), W_{1,4}(n)$.
- The number of representations of a natural number n as $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2$ is $12\sigma(n) - 36\sigma(n/3)$. What about similar quadratic forms of eight variables?
- We use modular forms and differential operators to give a general way to evaluate the convolution sums.

Modular forms

Modular forms

- Modular forms are special functions that are analogous to the trigonometric functions like sin, cos, tan... they are periodic functions.

- Modular forms have periodicity *plus enough extra symmetries that they are essentially unchanged under a large group of substitutions*.

- Symmetries allow to write modular forms as Fourier series $\sum_{n\geq 0} a_n q^n$ $(q = e^{2\pi iz})$.

- Coefficients a_n of modular forms contain information about *the number of representations of integers by positive definite binary quadratic forms, class numbers of quadratic fields, values of combinatorial functions, the number of points over the finite field F_p, p prime, of certain varieties such as elliptic curves, $K3$ surfaces, and Calabi-Yau threefolds*, etc.

- There are five fundamental operations in Mathematics: addition, subtraction, multiplication, division and *modular forms*.... Martin Eichler

Modular forms

Modular forms

- Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the complex upper half plane. Let $\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$
- Let f be a modular form of weight k for $SL_2(\mathbb{Z})$. i.e., f is a holomorphic function defined on the complex upper half-plane $\mathbb{H}$, which satisfies the following properties:

(1) $f(z+1) = f(z)$ and $f(-1/z) = (-z)^k f(z)$, or equivalently:

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \ \ for\ all\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

(2) $f(z)$ admits a Fourier expansion of the form:

$$f(z) = \sum_{n \geq 0} a(n) q^n, \qquad q = e^{2\pi i z}.$$

Modular forms

Modular forms

- Notation:
  M_k= Space of modular forms of weight k for $SL_2(\mathbb{Z})$
  S_k= Space of cusp forms (modular forms with $a_0 = 0$) of weight k for $SL_2(\mathbb{Z})$
- One can define modular form for congruence subgroup of level N of $SL_2(\mathbb{Z})$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N|c \right\}.$$
- Example. For even $k \geq 4$, the Eisenstein Series
$$E_k(z) = \frac{1}{\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$
  is a modular form of weight k.
- $E_4(z) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n$,
  $E_6(z) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n$.
  $E_8(z) = 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n$.
- In general, $E_k(z) = 1 - \dfrac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n$.

Modular forms

Modular forms

- If $f \in M_k$ and $g \in M_l$ then $fg \in M_{k+l}$.
- Example. $\Delta(z) = \frac{1}{1728}(E_4^3 - E_6^2) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$ is a cusp form of weight 12.
  ($\tau(n)$ is called the Ramanujan tau function.)
- $\tau(1) = 1, \tau(2) = -24, \tau(3) = 252, \tau(4) = -1472, \tau(5) = 4830, ..., \tau(10) = -115920$ $\tau(20) = -7109760, ..., \tau(29) = 128406630, \tau(30) = -29211840$.
- S. Ramanujan (1916) made the following conjectures about $\tau(n)$:

(1) $\tau(m)\tau(n) = \tau(mn)$ if $gcd(m,n) = 1$.

(2) $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$ for prime number p, $n \geq 1$.

(3) $|\tau(n)| \leq \sigma_0(n)n^{11/2}$.

The first two identities were proved by L. J. Mordell in 1921 and the third was proved by P. Deligne on 1974.

Lehmer's Conjecture

Lehmer's Conjecture

- Lehmer's Conjecture (1947) (still open)

$$\tau(n) \neq 0 \textit{ for all } n \geq 1.$$

- So far known (Derickx, van Hoeij, and Zeng (2013)):

$$\tau(n) \neq 0 \textit{ for } n < 816212624008487344127999.$$

- It is interesting to study identities and congruences related to $\tau(n)$.

Differential operator on modular forms

Differential operator on modular forms

- The derivative of a modular form is not a modular form.

  If we differentiate the transformation formula for f

  $$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

  w.r.t z, we get

  $$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f'(z) + kc(cz+d)^{k+1} f(z).$$

  Nevertheless, there are many interesting connections between differential operators and the theory of modular forms.

Differential operator on modular forms

Differential operator on modular forms

- If $f(z) \in M_k$, then
$$kf(z)f''(z) - (k+1)\{f'(z)\}^2 \in M_{2k+4}.$$

- R. A. Rankin (1956) considered the problem of finding all polynomials in the derivatives of $f(z)$ which are modular forms.

- Let $f \in M_k$ and $g \in M_\ell$. Then one has
$$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f'(z) + kc(cz+d)^{k+1} f(z)$$
and
$$g'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\ell+2} g'(z) + \ell c(cz+d)^{\ell+1} g(z)$$

Rankin-Cohen Brackets

Rankin-Cohen Brackets

- Suppose $F(z) = k(fg')(z) - \ell(f'g)(z)$. Then one checks easily that

$$F(z) \in M_{k+\ell+2}.$$

.

- Similarly,

$$\binom{k+1}{2} fg'' - (k+1)(\ell+1)f'g' + \binom{\ell+1}{2} f''g \in M_{k+\ell+4}.$$

- Theorem (Cohen, 1977): Let $f \in M_k$ and $g \in M_\ell$. Then for each non-negative integer n, define

$$[f,g]_n := \sum_{r=0}^{n} (-1)^r \binom{n+k-1}{n-r} \binom{n+\ell-1}{r} f^{(r)} \times g^{(n-r)}.$$

Then $[f,g]_n \in M_{k+\ell+2n}$, and for $n \geq 1$, $[f,g]_n \in S_{k+\ell+2n}$.

Rankin-Cohen Brackets

Rankin-Cohen Brackets

- Zagier (1994) described the operator studied by Cohen as the n-th Rankin-Cohen bracket and further discussed the algebraic properties of the brackets.
- van der Pol/ Niebur type identities for Ramanujan tau function: Using Rankin-Cohen bracket on modular form to we (with B. Ramakrishnan, 2011) prove several identities like:

$$\tau(n) = n^2\sigma_7(n) - 540\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m),$$

$$\tau(n) = n^4\sigma(n) - 24\sum_{m=1}^{n-1}(35m^4 - 52m^3n + 18m^2n^2)\sigma(m)\sigma(n-m).$$

Ramanujan-Serre Differential operator

Ramanujan-Serre Differential operator

- $E_2(z) = 1 - 24\sum_{n\geq 1}\sigma(n)q^n$ is a not a modular form

- $E_2\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^2E_2(z) + \dfrac{12c}{2\pi i}(cz+d)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

- E_2 is a quasimodular form of weight 2 of depth 1.

- Ramanujan (1916)
$$DE_2 = \frac{1}{12}(E_2^2 - E_4)$$
$$DE_4 = \frac{1}{3}(E_2E_4 - E_6),$$
$$DE_6 = \frac{1}{3}(E_2E_6 - E_8).$$

- Here $D = \dfrac{1}{2\pi i}\dfrac{d}{dz}$

Ramanujan-Serre Differential operator

Ramanujan-Serre Differential operator

- Use E_2 and differential operator D to create new modular forms.

- f is a modular form of weight k (of any level N) then

$$\delta_k(f) := Df - \frac{k}{12} E_2(z) f(z)$$

  is a modular form of weight $k+2$.

- $\delta_{12}(\Delta)$ is cusp form of weight 14, but there is no cusp form of weight 14 for $SL_2(\mathbb{Z})$ and hence $\delta_{12}(\Delta) = 0$ or equivalently $D\Delta = E_2\Delta$ and now we have (Ramanujan's recurrence) relation for $\tau(n)$

$$\tau(n) = \frac{24}{1-n} \sum_{m=1}^{n-1} \sigma(m)\tau(n-m).$$

Application

Application

- For positive integers a, b, define the differential operator $\delta_{k,a,b}$

$$\delta_{k,a,b}(f) := Df(bz) - \frac{k}{12} aE_2(z) f(bz)$$

which maps the space $M_k(M)$ to $M_{k+2}(N)$ (here $N = l.c.m(Mb, a)$).

- In particular, considering $f(z) = E_k(z) \in M_k$ with $gcd(a, b) = 1$

$$DE_k(bz) - \frac{k}{12} aE_2(z) E_k(bz) \in M_{k+2}(ab).$$

- Using a basis for the space $M_{k+2}(ab)$ we (with B. Ramakrishnan and A. K. Singh, 2022) evaluate $W_{a,b}^{1,k-1}(n)$.

$$W_{a,b}^{1,k-1}(n) = \frac{B_k}{2k}\sigma(n/a) + \left(\frac{1}{24} - \frac{n}{2ka}\right)\sigma_{k-1}(n/b) - \frac{B_k}{4ak^2}\sum_{i=1}^{\lambda_{k+2}(ab)} \alpha_i a_i(n),$$

where B_k is the k-th Bernoulli number and $a_i(n), 1 \leq i \leq \lambda_{k+2}(ab)$ are the n-th coefficients of a basis for the space $M_{k+2}(ab)$ with $\alpha(i)$ are some constants in $\mathbb{C}$.

Extension of Ramanujan's modular identities and application

Extension of Ramanujan's modular identities and application

- For $gcd(a, b) = 1$, the function

$$\frac{6}{b} DE_2(az) + \frac{6}{a} DE_2(bz) - E_2(az)E_2(bz) \in M_4(ab)$$

- As an application, we obtain the convolution sum $W_{a,b}(n)$ by using a basis for the space $M_4(ab)$.

$$W_{a,b}(n) = \frac{1}{24}\left(1 - \frac{6n}{b}\right)\sigma(n/a) + \frac{1}{24}\left(1 - \frac{6n}{a}\right)\sigma(n/b) - \frac{1}{576}\sum_{i=1}^{\lambda_4(ab)} \beta_i b_i(n),$$

where $b_i(n), 1 \leq i \leq \lambda_4(ab)$ are the n-th coefficients of a basis for the space $M_4(ab)$ with $\beta(i)$ are some constants in $\mathbb{C}$.

Theta Series and Computation of Modular forms

Theta Series and Computation of Modular forms

- Given a positive definite quadratic form, we consider the associated theta series which is a modular form (of some weight for some (congruence) subgroup of full modular group).
- Create a (suitable) basis/ online database L-function and Modular Forms Database (LMFDB)
- SAGE Math to compute modular form
- Find explicit formula for number of representation of a natural number n by quadratic form.
- $a_1x_1^2 + \cdots + a_kx_k^2$ for $k \geq 5$,
  $b_1(x_1^2 + x_1x_2 + x_2^2) + b_2(x_3^2 + x_3x_4 + x_4^2)$ and its duplications

Siegel modular forms of weight k degree n

Siegel modular forms of weight k degree n

- The Siegel upper half plane $\mathbb{H}_n$ of degree n of complex symmetric $n \times n$ matrices Z with positive-definite imaginary part.

- The symplectic group

$$Sp_{2n}(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in M_n(\mathbb{Z}), AB^t = BA^t, CD^t = DC^t, AD^t - BC^t = I_n \right\}$$

acts on $\mathbb{H}_n$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

- Consider holomorphic function $F : \mathbb{H}_n \to \mathbb{C}$ satisfying:

$$F((AZ + B)(CZ + D)^{-1}) = det(CZ + D)^k F(Z),$$

for all $Z \in H_n$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z})$.

Jacobi forms of weight k index m

Jacobi forms of weight k index m

- The Jacobi group $SL_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$ acts on $\mathbb{H} \times \mathbb{C}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \cdot (z, \tau) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

- Consider holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) = (c\tau + d)^k e^{m\left(\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} - \lambda^2\tau - 2\lambda z - \lambda\mu \right)} \phi(\tau, z)$$

for all $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in SL_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$.

- Jacobi forms arise as Fourier-Jacobi coefficients of Siegel modular form.

- Differential operators on Jacobi forms/ Siegel modular forms.

Some publications from Number Theory group

Some publications from Number Theory group

- Abhash K. Jha and Brundaban Sahu, Rankin-Cohen brackets on Siegel modular forms and special values of certain Dirichlet series, The Ramanujan J. (2017)
- Abhash K. Jha and Brundaban Sahu, Rankin-Cohen brackets on Jacobi forms of several variables and special values of certain Dirichlet series, Int. J. Number Theory (2019)
- R. Osburn, B. Sahu and A. Straub, Supercongruences for Sporadic Sequences, Proceedings of the Edinburgh Math. Soc. (2016)
- Jaban Meher, Sudhir Pujahari, K. Srinivas, Zeros of L-functions attached to modular forms of half-integral weight, Bull. London Math. Soc. (2017)
- Jaban Meher, Sudhir Pujahari, K. D. Shankhadhar, Zeros of L-functions attached to cusp forms of half-integral weight, Proc. American Math. Soc. (2019)
- K. Senthil Kumar, Order structure and topological properties of the set of multiple zeta values, International Math. Research Notices (2016)
- K. Senthil Kumar, Linear dependence of quasi-periods over the rationals, Comptes Rendus. Mathematique (2021)

Number Theory Workshop/ Conferences at NISER

Number Theory Workshop/ Conferences at NISER

- Discussion Meeting on Automorphic Forms (December 21-27, 2016)
- School and Workshop on Modular Forms and Black Holes (January 5–14, 2017)
- Workshop on Number Theory (November 30–December 6, 2018)
- Lecture series on Transcendence on Commutative Algebraic Groups (December 9–15, 2019)

Thank you

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