Dynamic And Fault Tolerant Algorithms

Manoj Gupta, IIT Gandhinagar

The Problem

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- Maintain solutions of graph theoretic / optimization problems more efficiently than recomputing from scratch

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Types of Changes

- Incremental/Decremental: only insertions/deletions of edges
- Fully dynamic: both insertions and deletions

Dynamic Graphs

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Performance Evaluation

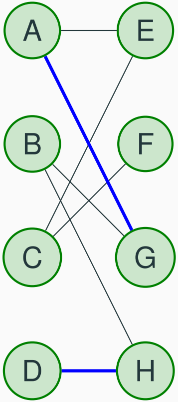
Update time: The time taken to Update the solution

Examples

- Connectivity
- Single source shortest path
- All pair shortest path
- Strongly connected components
- Minimum Spanning Tree
- Topological Sorting

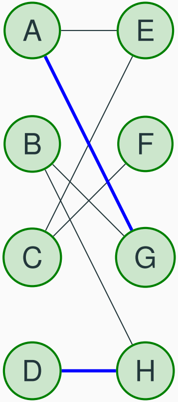
Some Definitions

- A matching in a graph is a set of edges M such that no two edges in M share a common endpoint



Some Definitions

- A matching in a graph is a set of edges M such that no two edges in M share a common endpoint
- We can find a $(1+\epsilon)$-approximate matching in a static unweighted graph in $O\left(\frac{m}{\epsilon}\right)$ time (Micali and Vazirani, 1980)



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Model

- At each update step an edge can be added or deleted from the graph
- Compute the matching quickly after each update

In this talk [G. and Peng (FOCS 2013)]

Maintain $(1+\epsilon)$-approximate maximum matching in $O\left(\frac{\sqrt{m}}{\epsilon^2}\right)$ update time

Key Idea

Can we find a smaller subgraph G' of G such that the size of the maximum matching in G' is same as the size of maximum matching in G?

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Answer

Yes : If you have a approximate vertex cover of the graph

$(1+\epsilon)$-Approximate matching

- Assume that we have an oracle access to the vertex cover V_{cover} at every update step

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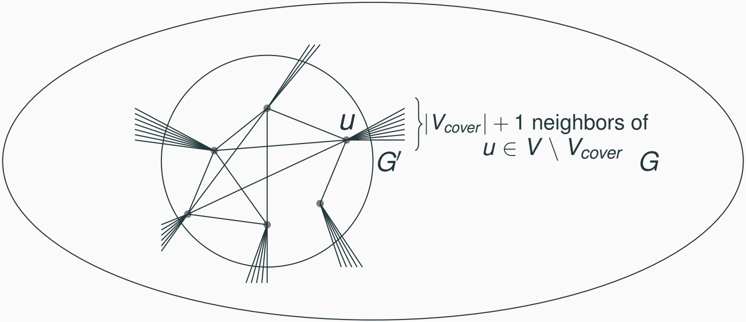
- Use the algorithm of Neiman and Solomon(STOC 2013): maintain 3/2-approximate matching in a dynamic graph
- Report all the vertices in the matching as V_{cover}

Core Graph

- Include all the edges within the vertex cover
- For each $u \in V_{cover}$, include at most $|V_{cover}| + 1$ neighbors outside the vertex cover



Theorem

The size of maximum matching in core graph G' is same as the size of maximum matching in G

Proof

Among all maximum matchings in G, let M' be one that uses the maximum number of edges in G'.

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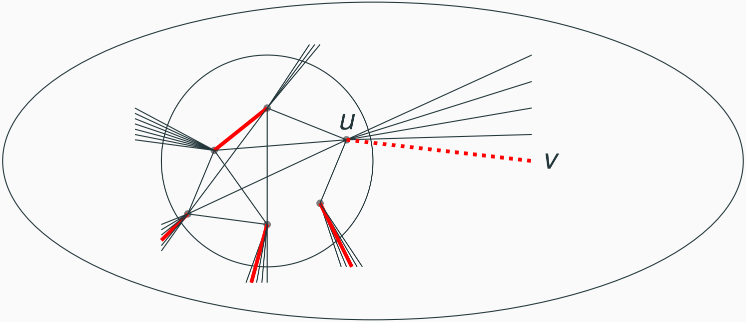

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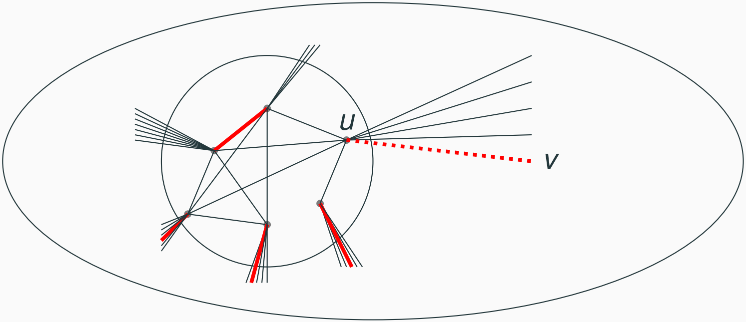

- By construction, u has $|V_{cover}|+1$ neighbors outside the vertex cover in G'.

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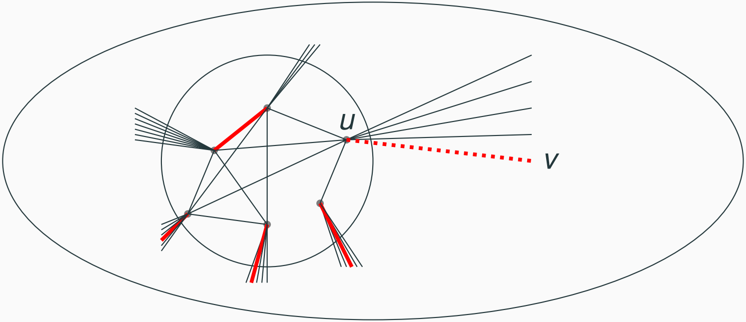
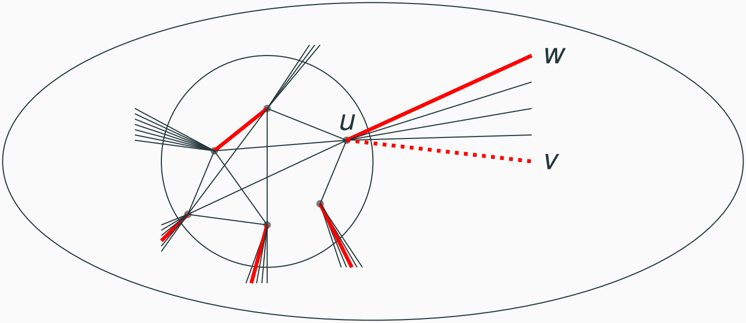


- By construction, u has $|V_{cover}|$+1 neighbors outside the vertex cover in G'.
- Atleast one of them is unmatched in M', because $|M'| \leq$ size of any vertex cover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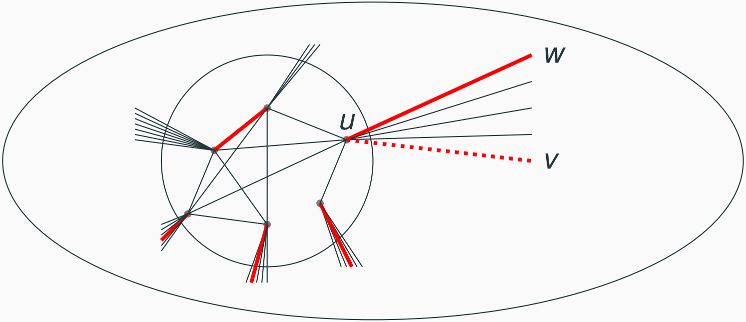
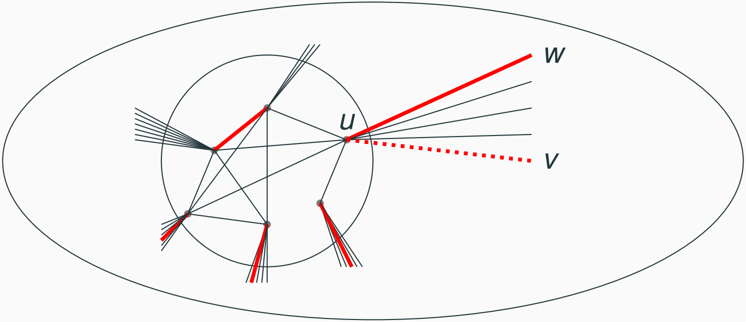


- $M'' \leftarrow M' \setminus (u, v) \cup (u, w)$



- $M'' \leftarrow M' \setminus (u, v) \cup (u, w)$
- M'' is a maximum matching and its intersection with G' is larger than that of M'
- A contradiction

Partial Algorithm

- Construct a core graph G' of G
- Find a $(1 + \epsilon/2)$ approximate matching M in G'

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- $|V_{cover}| \leq 2(1 + \epsilon/2)|M|$
- Size of G' is $min\{m, O(|M|^2)\}$
- Time to find a matching in G' is $O\Big(\frac{min\{m, |M|^2\}}{\epsilon}\Big)$

$(1+\epsilon)$-approximate matching

Algorithm

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- M can reduce by atmost 1 wrt maximum matching after each update step

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Analysis

- M can reduce by atmost 1 wrt maximum matching after each update step
- After $\epsilon|M|/2$ steps, the matching M is $(1+\epsilon)$-approximate

Running time

- If $|M| \geq \sqrt{m}$, the amortized update time is $O\Big(\frac{m\epsilon^{-1}}{\epsilon|M|}\Big) = O\Big(\frac{\sqrt{m}}{\epsilon^2}\Big)$ $\qquad (min\{m, |M|^2\})$

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- If $|M| < \sqrt{m}$, the amortized update time is $O\Big(\frac{|M|^2\epsilon^{-1}}{\epsilon|M|}\Big) = O\Big(\frac{|M|}{\epsilon^2}\Big) = O\Big(\frac{\sqrt{m}}{\epsilon^2}\Big) \qquad (min\{m, |M|^2\})$

Theorem

Maintain $(1+\epsilon)$-approximate maximum matching in $O\left(\frac{\sqrt{m}}{\epsilon^2}\right)$ update time

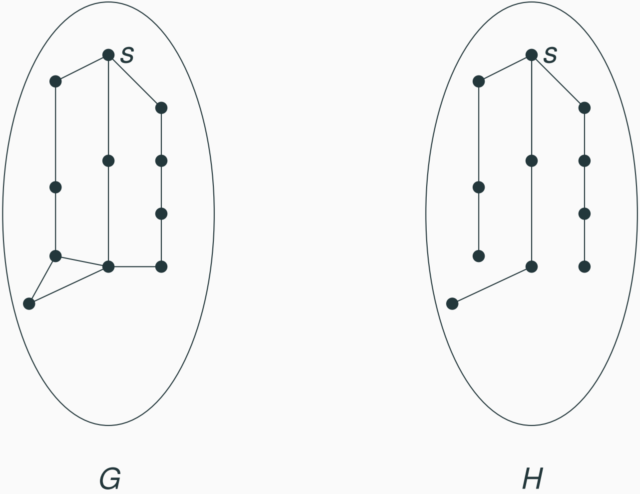
Make your own problem

- Incremental, Decremental or Fully Dynamic
- Unweighted or Weighted graphs
- Approximate matching or maximum matching
- Randomized or deterministic
- Worst case update time or Amortized running time
- Directed or Undirected graph

Fault Tolerant Algorithms

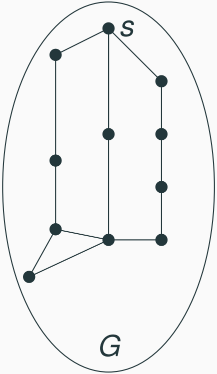
A Fault Tolerant System continues to perform at a desired level in spite of failures in some of its components.

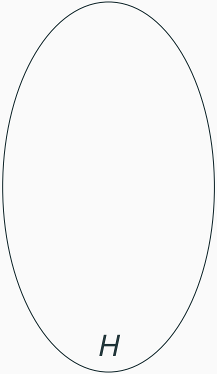
Fault Tolerant Subgraph Problem



Find a subgraph H of G such that the shortest path from s to all other vertices are preserved in H.

Fault Tolerant Subgraph Problem





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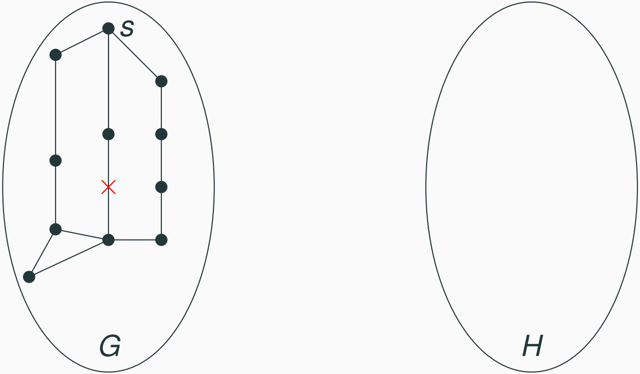
Fault Tolerant Subgraph Problem



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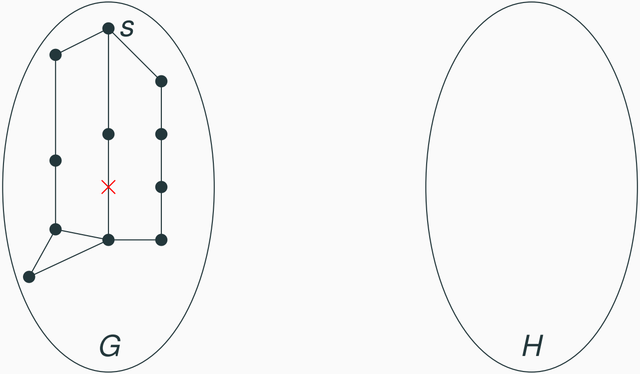

- Find a subgraph H of G such that the shortest path from s to all other vertices avoiding a single edge are preserved in H.
- Parter and Peleg [ESA 2013] showed that $O(n^{3/2})$ edges are both sufficient and necessary.

Fault Tolerant Algorithm

Input $\Longrightarrow$ **Data-Structure**

- Preprocess the input to build a data-structure.
- Preprocessing is free.

Input $\Longrightarrow$ **Data-Structure** $\Longleftrightarrow$ **Query Algorithm**

- Design a query algorithm that will use your data-structure to answer queries efficiently.

Fault Tolerant Algorithm

Input $\Longrightarrow$ **Data-Structure** $\Longleftrightarrow$ **Query Algorithm**

- Given a graph G design a data-structure that can answer the following query: find the length of shortest path from a source s to v where $v \in V$.

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- Given a graph G design a data-structure that can answer the following query: find the length of shortest path from a source s to v where $v \in V$.
- Store the distances from s in $O(n)$ space, so that queries can be answered in $O(1)$ time.

Our problem

- Given an undirected and unweighted graph G, design a data-structure that can find the shortest path from a source node s to any destination node avoiding a single edge.

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- Given an undirected and unweighted graph G, design a data-structure that can find the shortest path from a source node s to any destination node avoiding a single edge.
- Formally, the query algorithm should answer the following query quickly, $\textsc{Query}(s, t, e)$: Find the length of the shortest path from s to $t \in V$ avoiding the edge e.
- Such a (data-structure + query algorithm) is known as distance oracle.

In this talk [G. and Singh, ICALP 2018]

We present a distance oracle of size $\tilde{O}(n^{3/2})$ that can answer queries in $\tilde{O}(1)$ time.

1. Sample a set of terminals $\mathcal{T}$ of size $\tilde{O}(\sqrt{n})$ vertices.

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2. With a high probability, on any *st* path, there exists a vertex $t_s \in \mathcal{T}$ such that $|t_s t| = \tilde{O}(\sqrt{n})$.

Near case: $e \in t_s t$

Far case: $e \in s t_s$

The Near Case

1. Store all replacement paths that avoid edges in $t_s t$.
2. Number of shortest paths stored (for a fixed t) is $|t_s t| = \tilde{O}(\sqrt{n})$

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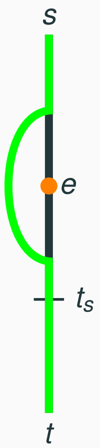
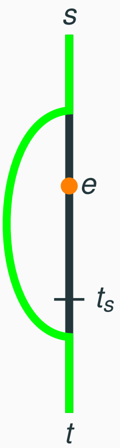
1. Store all replacement paths that avoid edges in $t_s t$.
2. Number of shortest paths stored (for a fixed t) is $|t_s t| = \tilde{O}(\sqrt{n})$
3. The size of the data-structure for a fixed t is $\tilde{O}(\sqrt{n})$.
4. The total size of the data-structure is $\tilde{O}(n^{3/2})$.

The Far Case



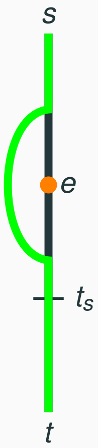
Replacement path passes through t_s



Replacement path avoids t_s

Replacement path passes through t_s

1. Store the length of the shortest path from s to $t_s \in \mathcal{T}$ avoiding each edge on st_s path.
2. The space taken = # terminals $\times$# edges on st_s path
   = $\tilde{O}(\sqrt{n}) \times n = \tilde{O}(n^{3/2})$



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3. Store the length of the shortest path from $t_s \in \mathcal{T}$ to $t \in V$.
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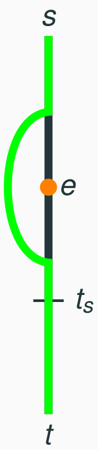

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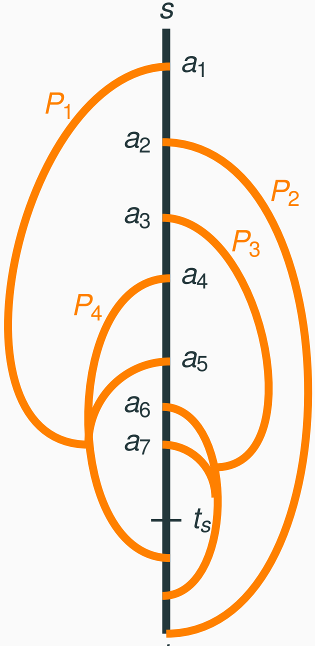
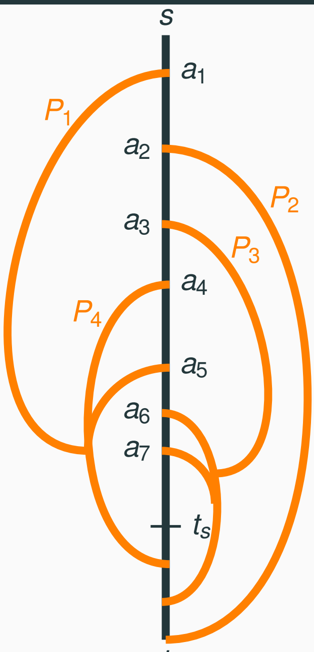


The replacement path avoids t_s

Main Technical Result

The total number of replacement paths from s to t that avoid t_s is $O(\sqrt{n})$.



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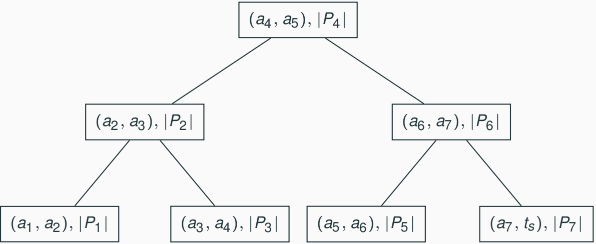

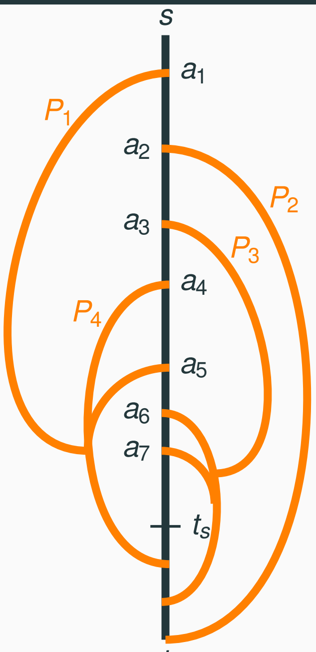

Size of our last data-structure

Since the size of the BST is $O(\sqrt{n})$ (for a fixed t), the total size of the data-structure is $O(n^{3/2})$.

Main Theorem

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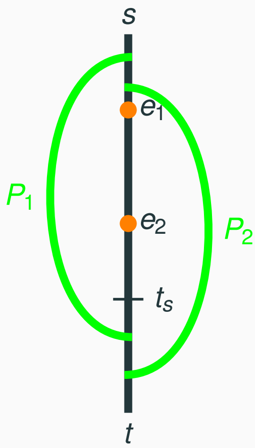
Rest of the talk

The total number of replacement paths from s to t that avoid t_s is $O(\sqrt{n})$.

The replacement path avoids t_s

Few basic observations



The replacement path avoids t_s

Few basic observations

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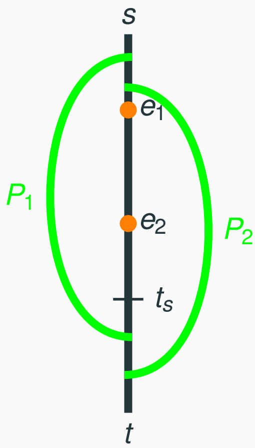
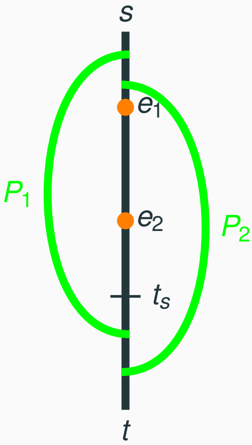


The replacement path avoids t_s

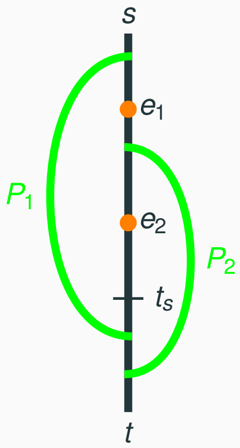
Few basic observations

- Is the picture correct?
- No, because if $|P_1| \leq |P_2|$, then the replacement path that avoids e_2 is also P_1.



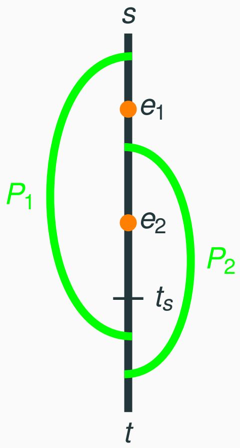
The replacement path avoids t_s

- Is the picture in the right correct?
- No, because if $|P_1| \leq |P_2|$, then the replacement path avoid e_2 is also P_1.
- $\mathcal{P}_1$: The lower replacement path will pass through the edge avoided by the upper replacement path.



The replacement path avoids t_s

- Can $|P_2| \geq |P_1|$?



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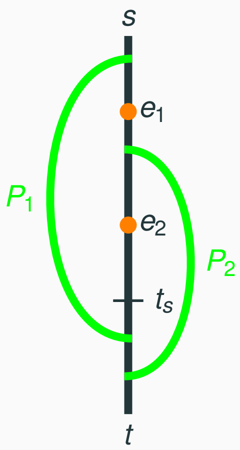

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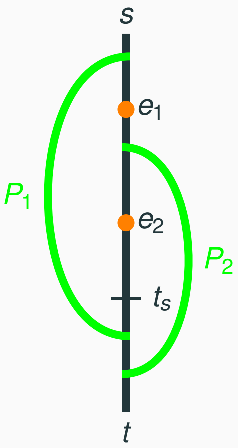
- No, because if $|P_2| \geq |P_1|$, then the replacement path avoiding e_2 is P_1.
- $\mathcal{P}_2$: The lower replacement path has length strictly less than the upper replacement path.



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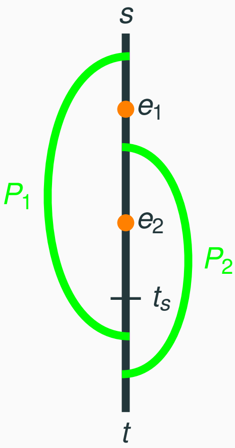
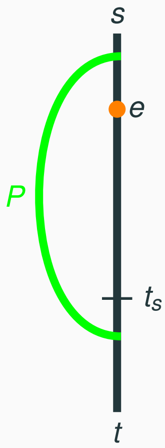
- No, because if $|P_2| \geq |P_1|$, then the replacement path avoiding e_2 is P_1.
- $\mathcal{P}_2$: The lower replacement path has length strictly less than the upper replacement path.
- Corollary: The length of these paths are distinct.

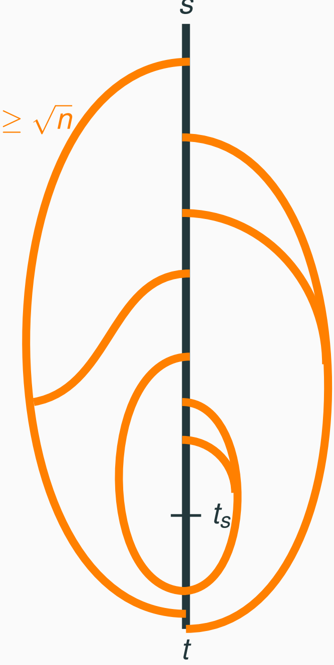


Some Definitions

- Detour of a replacement path.
- Green path or formally $P \setminus st$



- Process replacement paths from top to bottom.
- Try to associate $\sqrt{n}$ unique vertices of the detour with each replacement path



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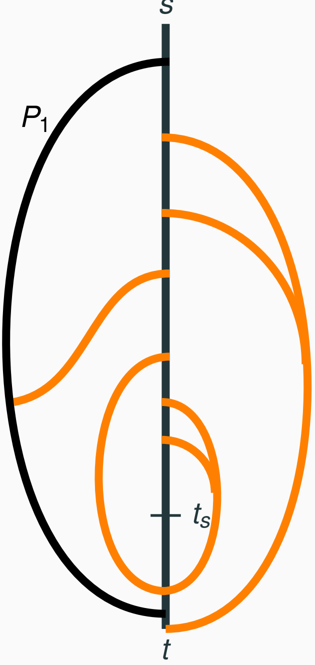

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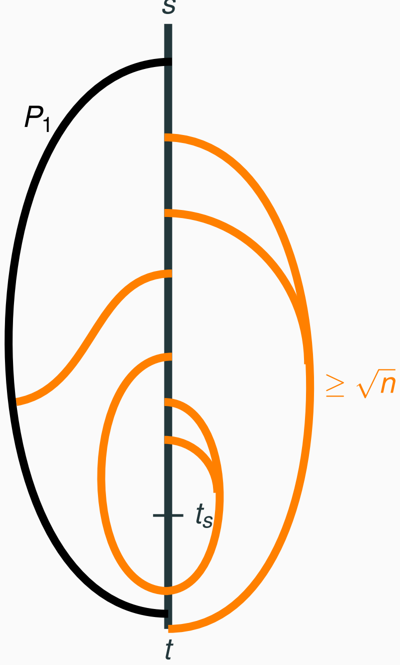

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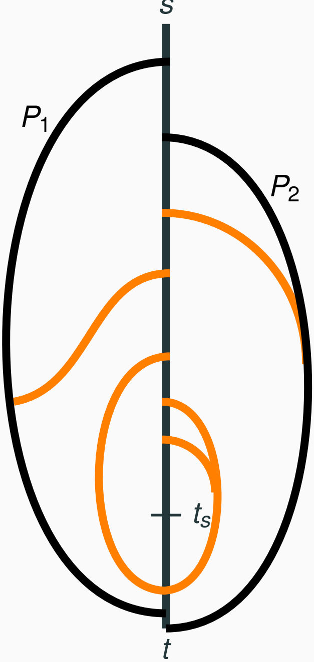

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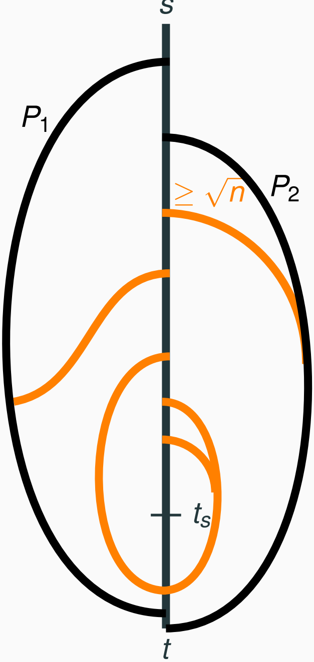

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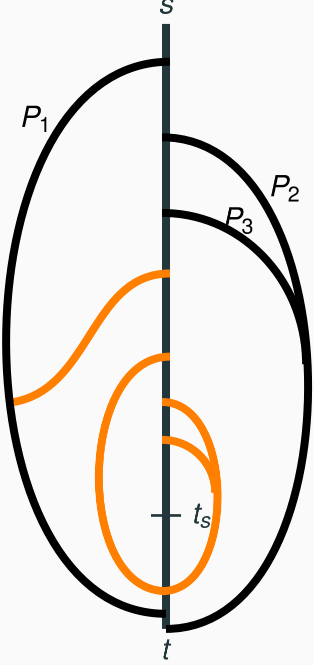
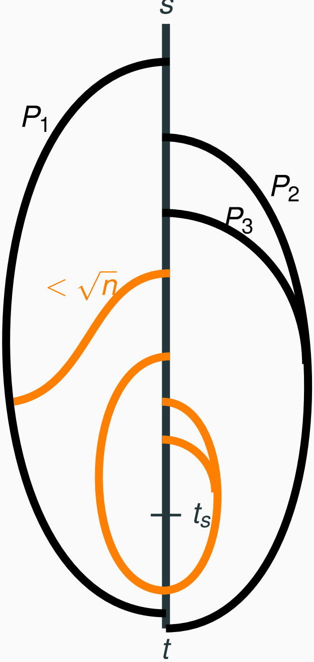
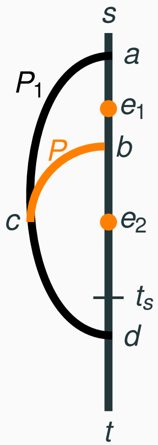


- Process replacement paths from top to bottom.
- Try to associate $\sqrt{n}$ unique vertices of the detour with each replacement path



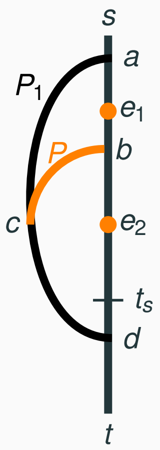
Bad case when $|bc| < \sqrt{n}$

- $|P_1| = |sa| + |ac| + |cd| + |dt|$



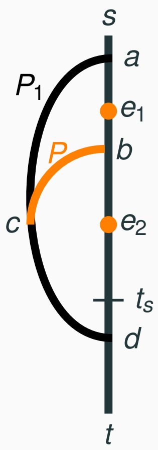
Bad case when $|bc| < \sqrt{n}$

- $|P_1| = |sa| + |ac| + |cd| + |dt|$
- But there is another path from s to t that avoids e_1.



Bad case when $|bc| < \sqrt{n}$

- $|P_1| = |sa| + |ac| + |cd| + |dt|$
- But there is another path from s to t that avoids e_1.
- |sa| + |ac| + |cb| + |bt|



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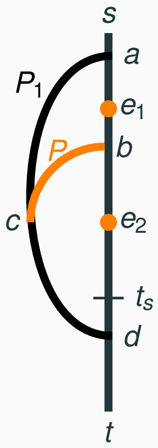
- $|P_1| = |sa| + |ac| + |cd| + |dt|$
- But there is another path from s to t that avoids e_1.
- |sa| + |ac| + |cb| + |bt|
- Why is this a valid path avoiding e_1?.



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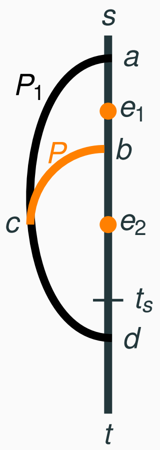
- Why is this a valid path avoiding e_1?.
  - cb is the part of the detour. So, it cannot pass through e_1.
  - Regarding bt, by $\mathcal{P}_1$, lower replacement path (P) passes through the edge avoided by the higher replacement path. So, b lies below e_1. Thus, bt doest not contain e_1.



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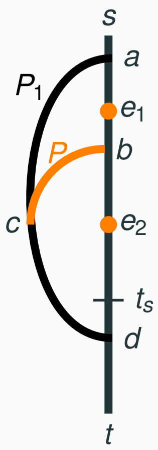
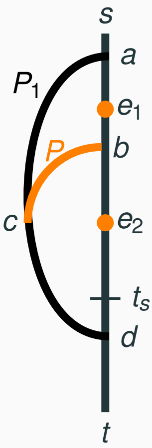
    - Regarding bt, by $\mathcal{P}_1$, lower replacement path (P) passes through the edge avoided by the higher replacement path. So, b lies below e_1. Thus, bt doest not contain e_1.
- Since this path was not chosen by our algorithm as the replacement path, its length must be $>$ the length of P_1.



Bad case when $|bc| < \sqrt{n}$

- $|sa| + |ac| + |cd| + |dt| < |sa| + |ac| + |cb| + |bt|$



Bad case when $|bc| < \sqrt{n}$

- $|sa| + |ac| + |cd| + |dt| < |sa| + |ac| + |cb| + |bt|$
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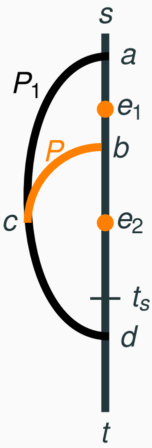

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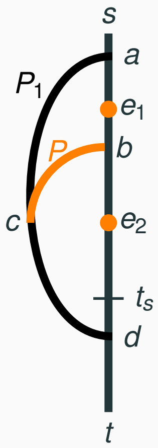
- $|sa| + |ac| + |cd| + |dt| < |sa| + |ac| + |cb| + |bt|$
  $\implies |cd| + |dt| < |cb| + |bt|$
  $\implies |bc| + |cd| + |dt| < 2|cb| + |bt|$



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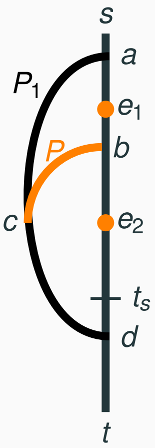
  $\implies |bc| \ + |cd| + |dt| < 2\sqrt{n} + |bt|$



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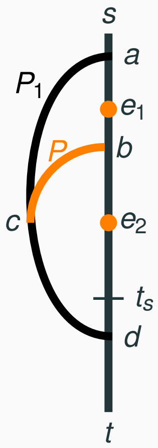
  $\implies |bc| + |cd| + |dt| < 2\sqrt{n} + |bt|$
- On the left hand side we have a replacement path from b to t avoiding e_2.



Bad case when $|bc| < \sqrt{n}$

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  $\implies |cd| + |dt| < |cb| + |bt|$
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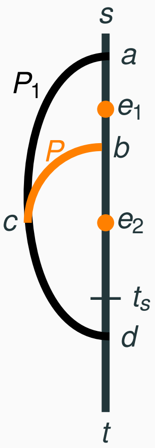
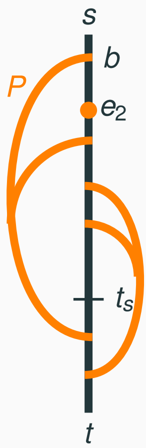
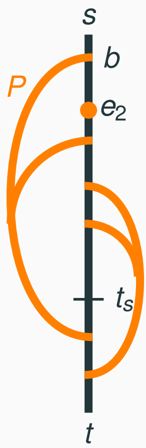
  $\implies |bc| + |cd| + |dt| < 2\sqrt{n} + |bt|$
- On the left hand side we have a replacement path from b to t avoiding e_2.
- A good property of this replacement path is that its length is just $2\sqrt{n}$ greater than bt. We now exploit this property.

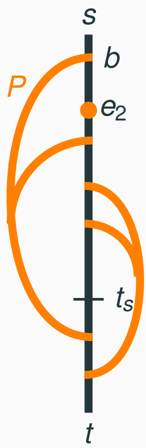



s
b
P
e_2
t_s
t

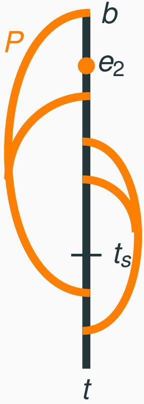
- By Property $\mathcal{P}_1$, all these lower replacement path pass through the edge avoided by P, that is e_2.



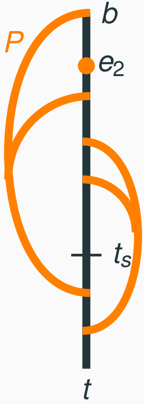
- By Property $\mathcal{P}_1$, all these lower replacement path pass through the edge avoided by P, that is e_2.
- We can thus assume that these paths are starting from b.



- By Property $\mathcal{P}_1$, all these lower replacement path pass through the edge avoided by P, that is e_2.
- We can thus assume that these paths are starting from b.
- By Property $\mathcal{P}_2$, the lower replacement path have length strictly less than the upper replacement path, that is P.



- By Property $\mathcal{P}_1$, all these lower replacement path pass through the edge avoided by P, that is e_2.
- We can thus assume that these paths are starting from b.
- By Property $\mathcal{P}_2$, the lower replacement path have length strictly less than the upper replacement path, that is P.
- The corollary of $\mathcal{P}_2$ states that length of these paths are distinct.



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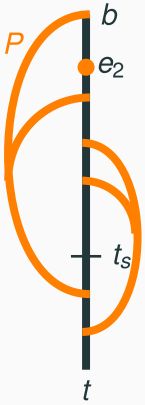
- Length of these path strictly lie in the range $[|bt|, |bt| + 2\sqrt{n}]$



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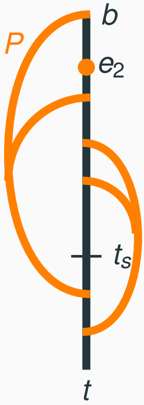

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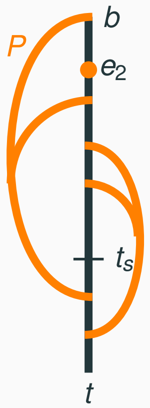
- The corollary of $\mathcal{P}_2$ states that length of these paths are distinct.
- Length of these path strictly lie in the range $[|bt|, |bt| + 2\sqrt{n}]$ $=O(\sqrt{n})$



Main Technical Result

The total number of replacement paths from s to t that avoid t_s is $O(\sqrt{n})$.

- We extend the above result to multiple sources.
- The extension, though technically involved, uses the strategy shown in this talk.

Open Problems

- What happens for two edge faults?
- For any general k edge faults?
- Fault tolerant all pair shortest path.

Thank You